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Small-Signal Modeling of the Boost Converter Operated in CM

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IEEE Senior Member

Course Agenda

- ❑ The PWW Switch Concept
- ❑ Small-Analysis in Continuous Conduction Mode
- ❑ Small-Signal Response in Discontinuous Mode
- ❑ EMI Filter Output Impedance
- ❑ Cascaded Converters Operation

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- ❑ **The PWW Switch Concept**
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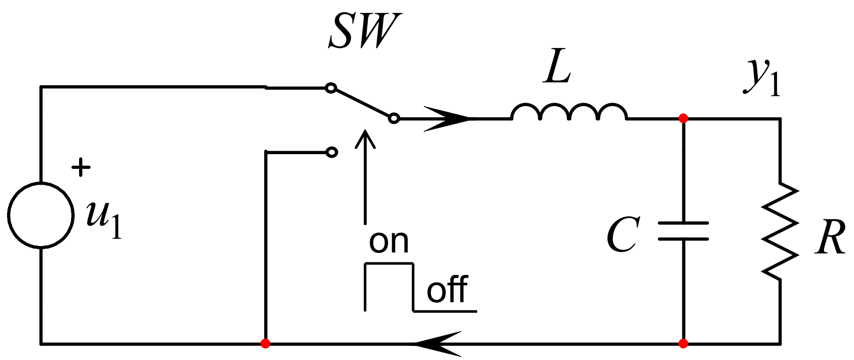
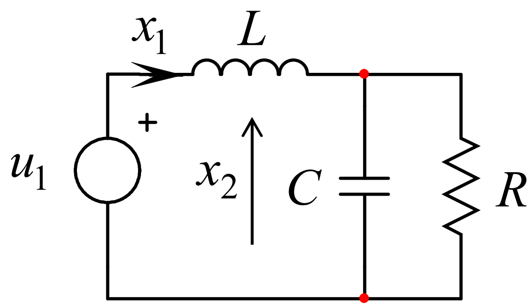
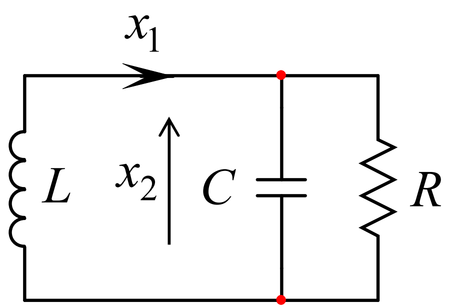
Non-Linearity in a Switching Converter

- A switching converter is ruled by linear equations…





during DT_{sw}

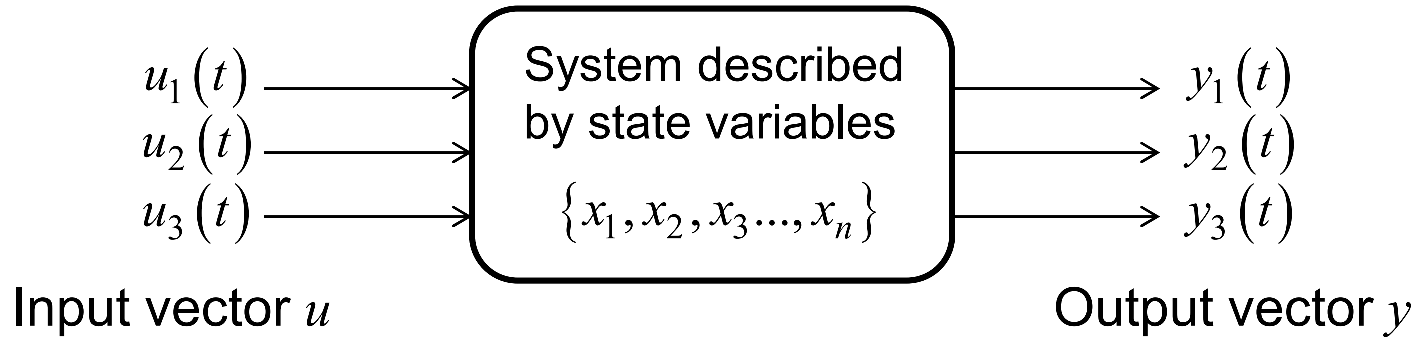


during $(1-D)T_{sw}$

- …combining so-called state variables

State Variables

- State variables describe the mathematical state of a system
- n state variables for n independent storage elements
- knowing variables state at t_0 helps compute outputs for $t > t_0$



- x_1 is the inductor current and x_2 is the capacitor voltage
- Differentiation gives the state variable rate of change

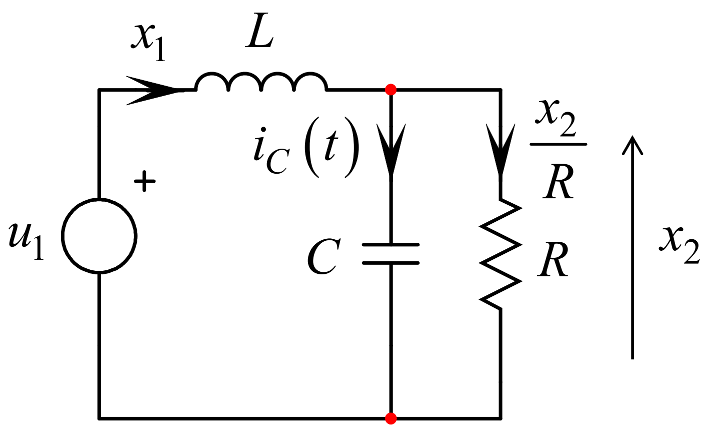
$$x_1 = i_L(t) \qquad \dot{x}_1 = \frac{di_L(t)}{dt}$$

Predict future system state

$$x_2 = v_C(t) \qquad \dot{x}_2 = \frac{dv_C(t)}{dt}$$

Describe the System During the On-Time

- Observe the system during the on-time duration or dT_{SW}:



$$i_C(t) = C\frac{dv_C(t)}{dt} = C\dot{x}_2 = x_1 - \frac{1}{R}x_2$$

$$u_1 = L\frac{di_L(t)}{dt} + v_C(t) = L\dot{x}_1 + x_2$$

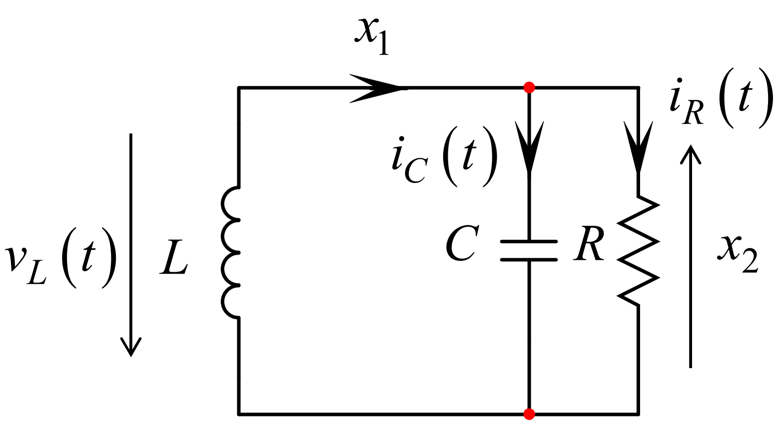
$$\left.\begin{aligned} \dot{x}_1 &= -\frac{1}{L}x_2 + \frac{1}{L}u_1 \\ \dot{x}_2 &= \frac{1}{C}x_1 - \frac{1}{RC}x_2 \end{aligned}\right\} \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/L & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

State coefficients

Source coefficients

Describe the System During the Off-Time

- Repeat the exercise during the off-time duration or $1\text{-}dT_{SW}$



$$v_L(t) = -x_2 \quad v_L(t) = L\dot{x}_1$$

$$\Rightarrow \quad x_2 = -L\dot{x}_1$$

$$i_R(t) = x_1 - i_C(t) = x_1 - C\dot{x}_2$$

$$x_2 = i_R(t)R$$

$$\Rightarrow \quad x_2 = R(x_1 - C\dot{x}_2)$$

$$\Rightarrow \quad \left.\begin{aligned} \dot{x}_1 &= 0 - \frac{1}{L}x_2 \\ \dot{x}_2 &= \frac{1}{C}x_1 - \frac{1}{RC}x_2 \end{aligned}\right\} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

State coefficients

Source coefficients

Make it Fit the State Equation Format

- Arrange expressions to make them fit the format:

$$\dot{x} = \mathbf{A}x(t) + \mathbf{B}u(t)$$ State equation

on-time network

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/L & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \mathbf{A}_1 = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \mathbf{B}_1 = \begin{bmatrix} 1/L & 0 \\ 0 & 0 \end{bmatrix}$$

off-time network

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \mathbf{A}_2 = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \mathbf{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- How do we link matrixes $\mathbf{A}_1$ and $\mathbf{A}_2$, $\mathbf{B}_1$ and $\mathbf{B}_2$?
- We smooth the discontinuity by weighting them by D and $1\text{-}D$

$$\dot{x} = \left[\mathbf{A}_1 D + \mathbf{A}_2 (1 - D)\right] x(t) + \left[\mathbf{B}_1 D + \mathbf{B}_2 (1 - D)\right] u(t)$$

The State-Space Averaging Method (SSA)

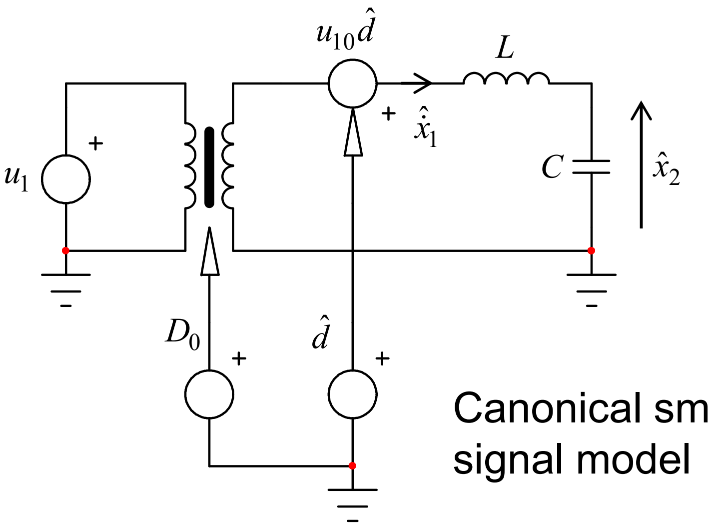
- We now have a continuous large-signal equation
- We need to linearize it via perturbations

$$D = D_0 + \hat{d} \qquad x = x_0 + \hat{x} \qquad u = u_0 + \hat{u}$$

$$\dot{x} = \left[\mathbf{A}_1 D + \mathbf{A}_2 (1 - D)\right] x(t) + \left[\mathbf{B}_1 d + \mathbf{B}_2 (1 - D)\right] u(t)$$

$$\dot{\hat{x}}_1 = \frac{1}{L}\hat{x}_2 + \frac{D_0}{L}\hat{u}_1 + \frac{\hat{d}}{L}u_{10}$$

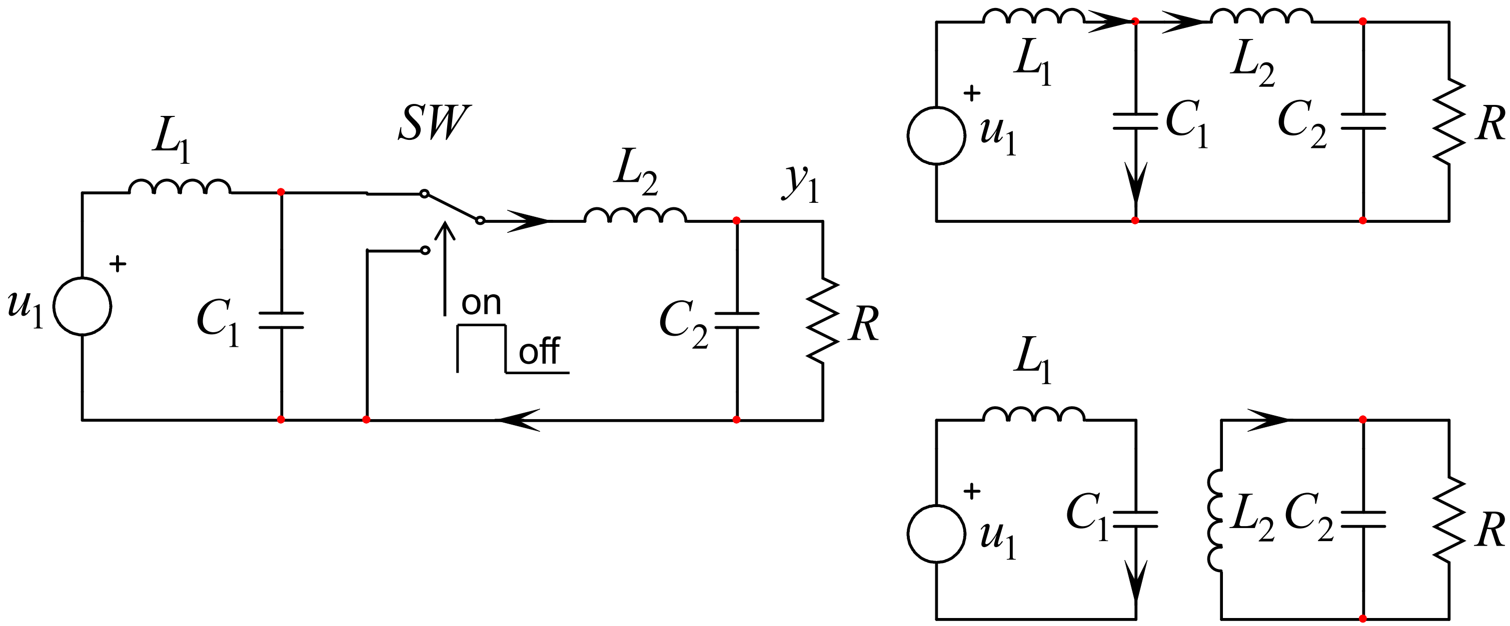
$$\dot{\hat{x}}_2 = \frac{1}{C}\hat{x}_1 - \frac{1}{RC}\hat{x}_2$$



Canonical small-signal model

The State-Space Averaging Method (SSA)

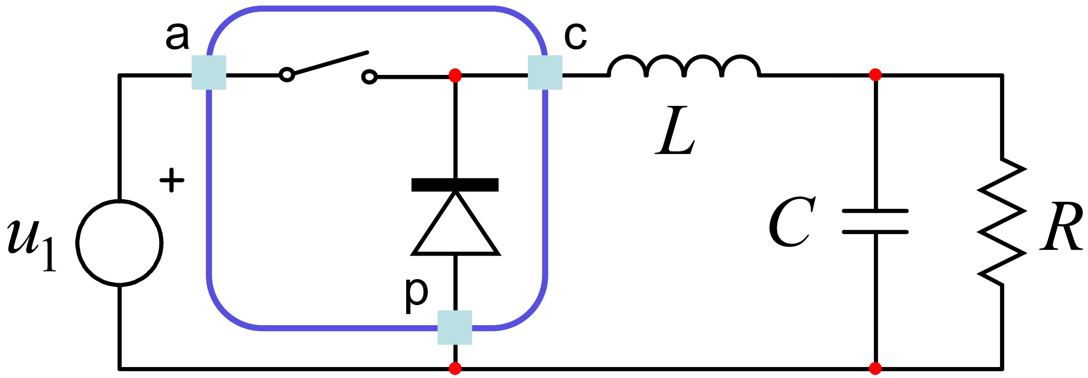
- ❑ SSA was applied to switching converters by Dr Ćuk in 1976
- ❑ It is a long, painful process, manipulating numerous terms
- ❑ What if you add an EMI filter for instance?



- ❑ 4 state variables and you have to re-derive all equations!

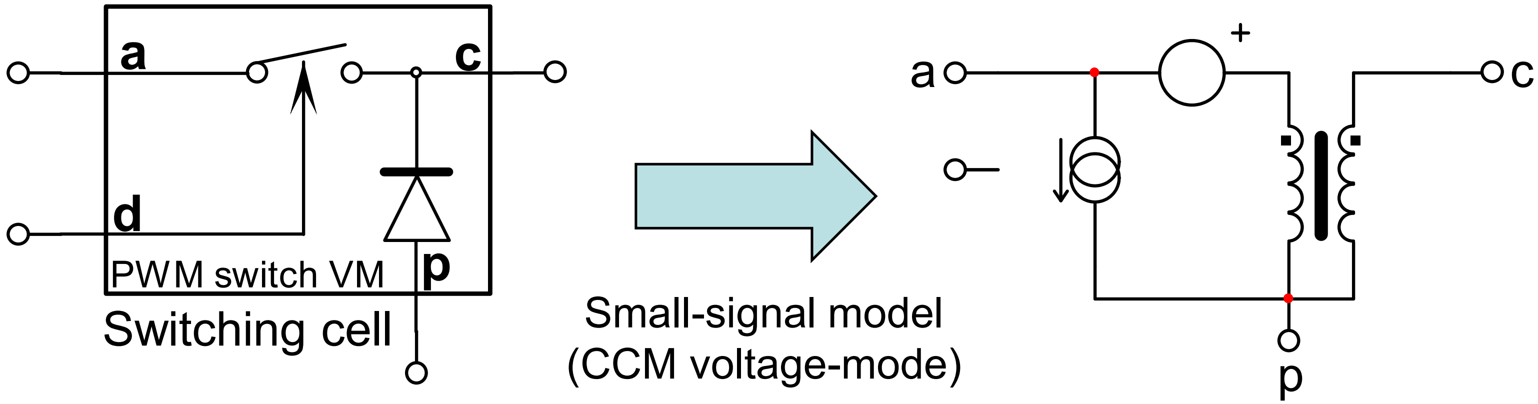
The PWM Switch Model in Voltage Mode

- We know that non-linearity is brought by the switching cell



a: active
c: common
p: passive

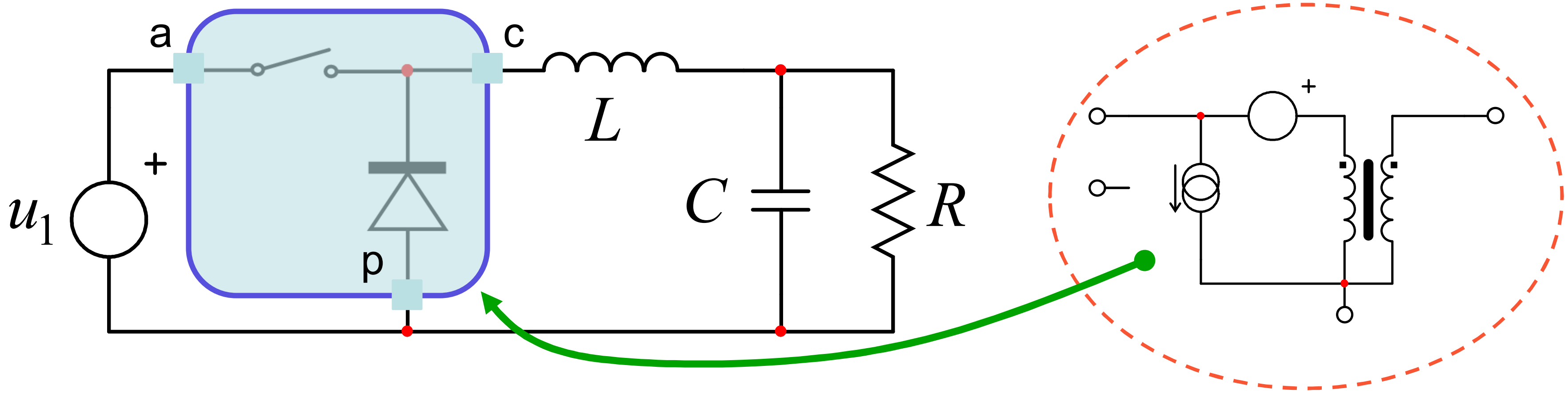
- Why don't we linearize the cell alone?



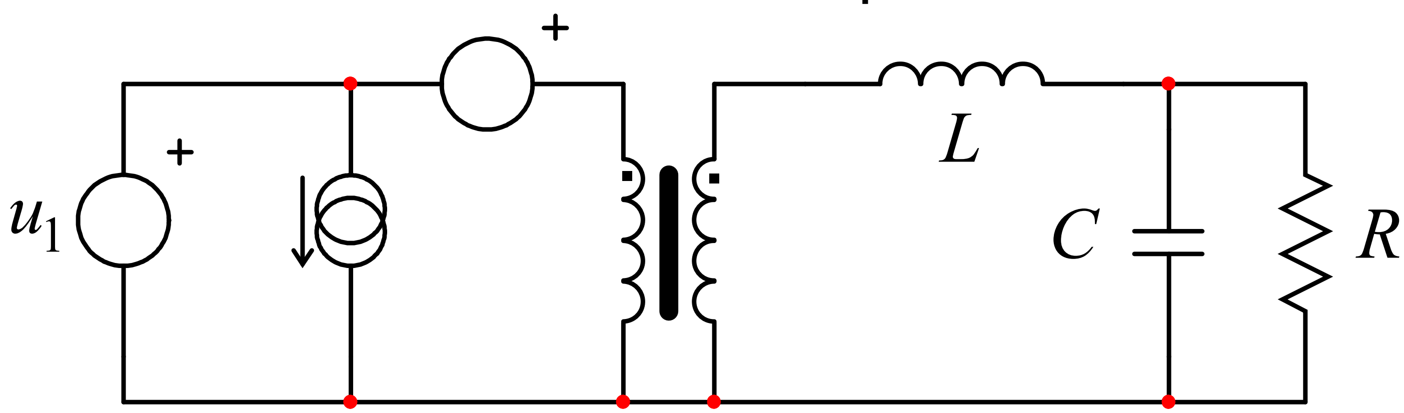
V. Vorpérian, "Simplified Analysis of PWM Converters using Model of PWM Switch, parts I and II" IEEE Transactions on Aerospace and Electronic Systems, Vol. 26, NO. 3, 1990

Replace the Switches by the Model

❑ Like in the bipolar circuit, replace the switching cell…

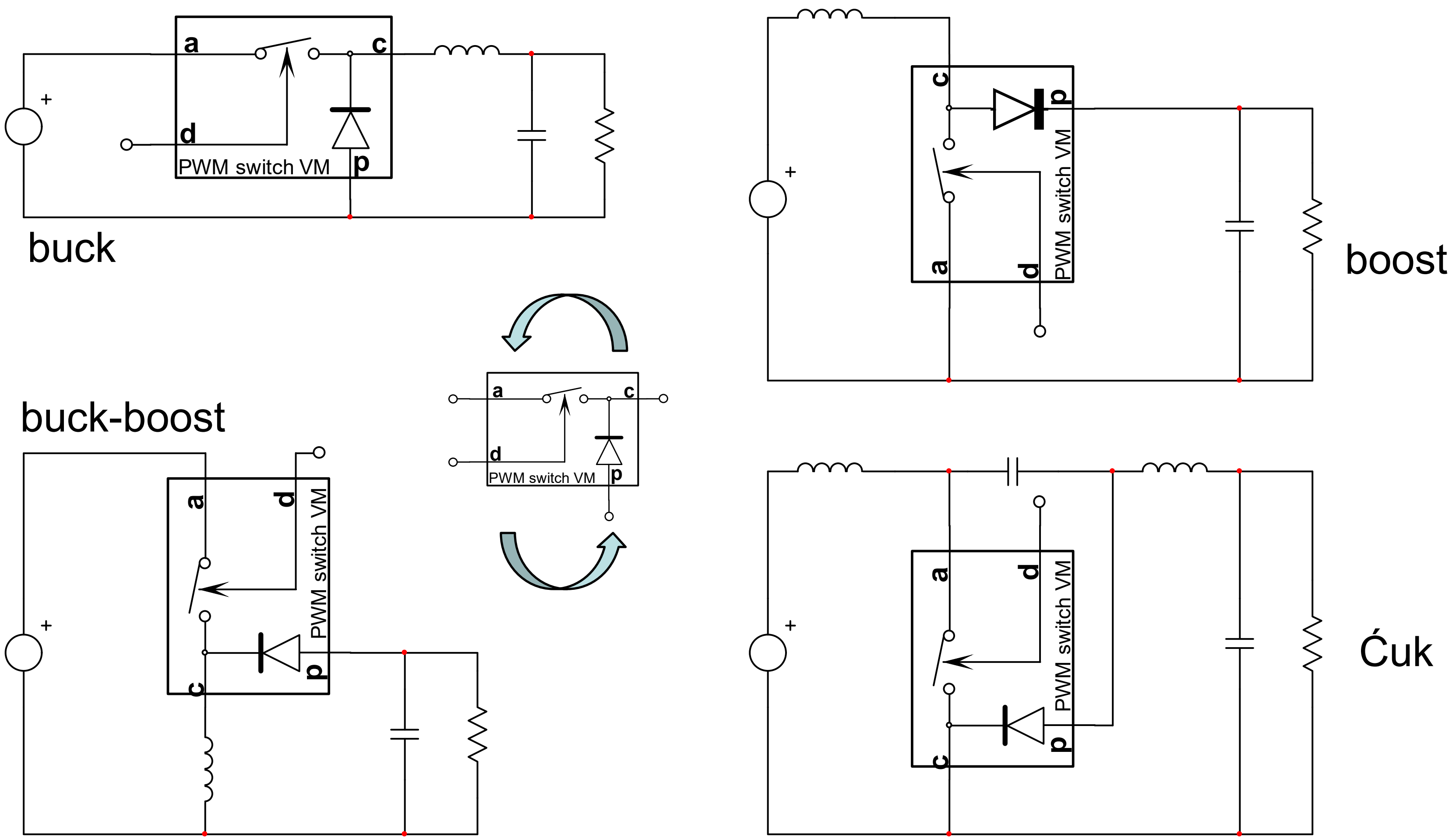


❑ …and solve a set of linear equations!



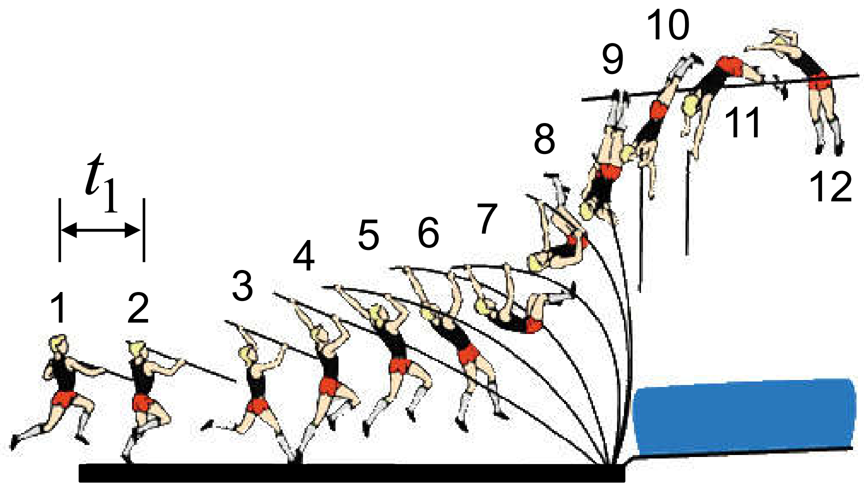
An Invariant Model

- ❑ The switching cell made of two switches is everywhere!



Smoothing the Discontinuity

- A CRT TV displays frames at a certain rate, 50 per second



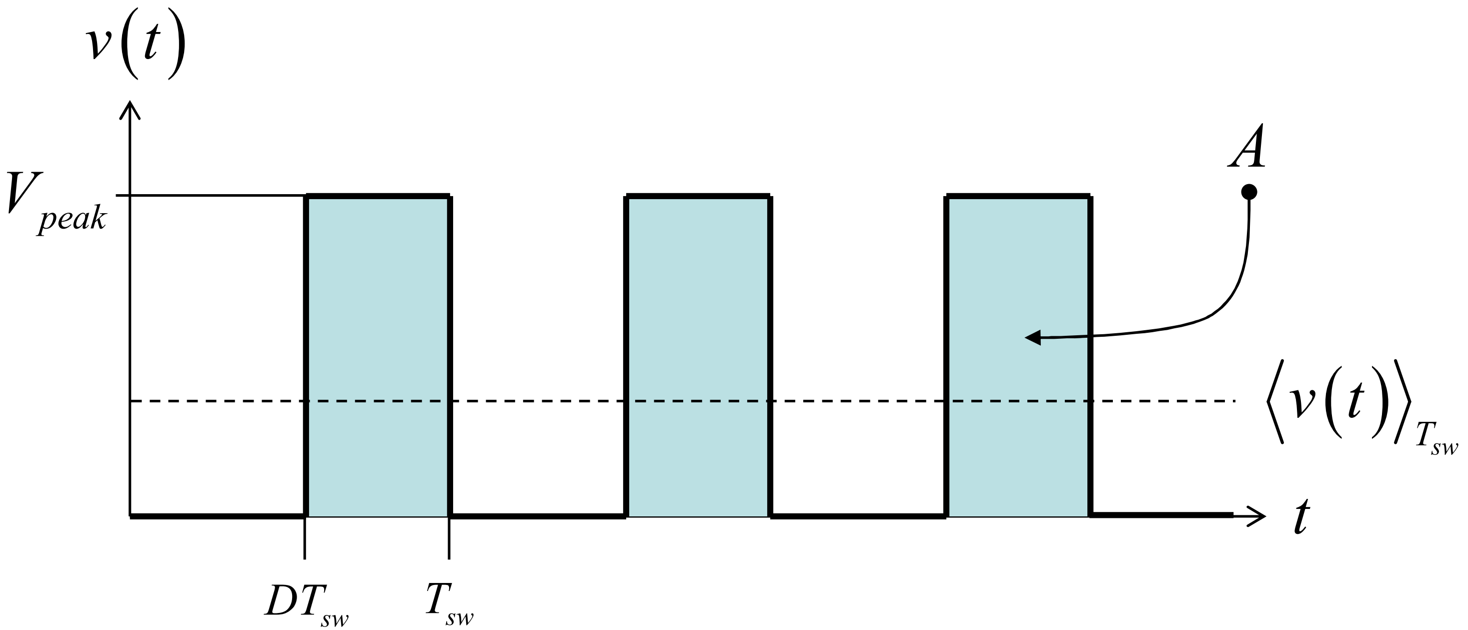
- The optic nerve time constant is larger than an interval
- A succession of discrete events is seen as continuous

Low-frequency filtering
Integration

See "phi phenomena", www.wikipedia.org

Averaging Waveforms

- The keyword in the PWM switch is *averaging*

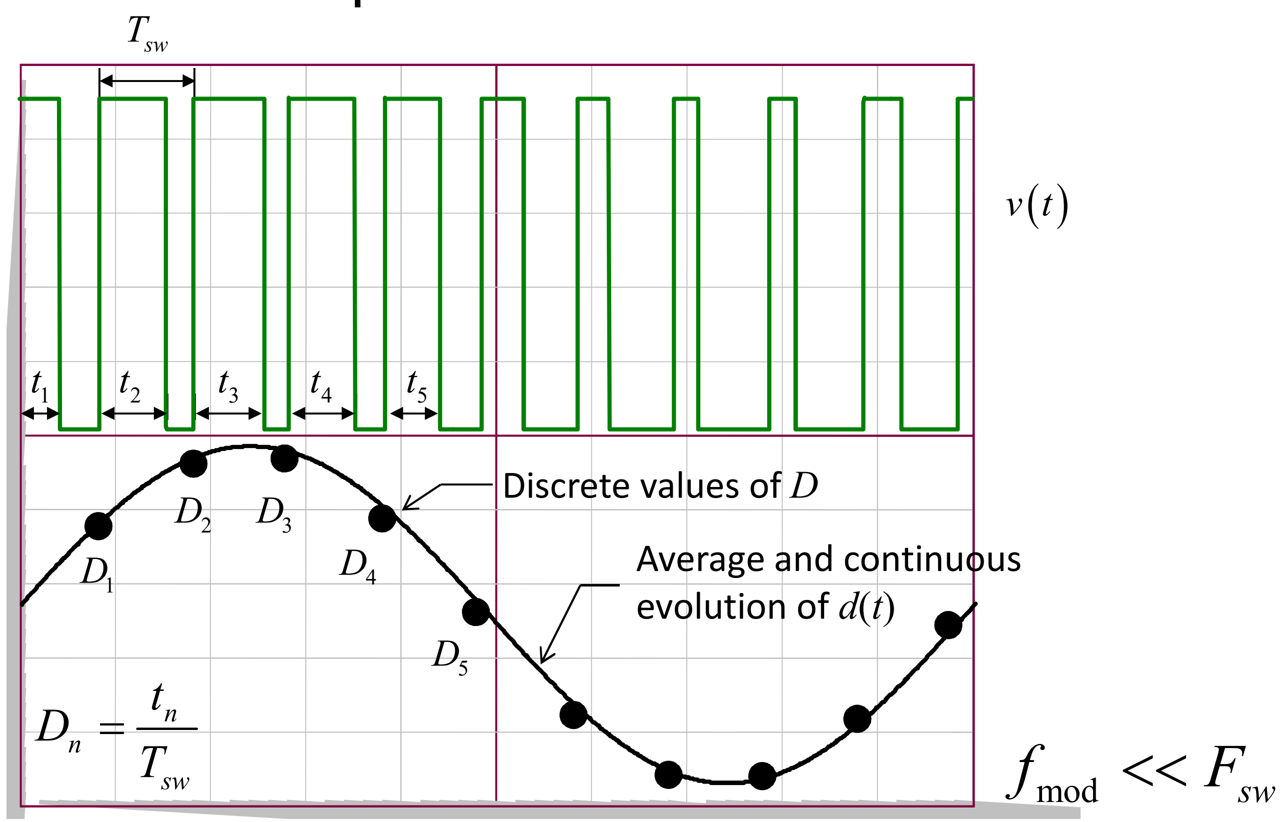


$$\langle v(t) \rangle_{T_{sw}} = \frac{1}{T_{sw}} \int_{0}^{T_{sw}} v(t) \cdot dt = \frac{1}{T_{sw}} \int_{DT_{sw}}^{T_{sw}} V_{peak} \cdot dt = V_{peak} (1 - D) = V_{peak} D'$$

- The resulting function is continuous in time

From Steps to a Continuous Function

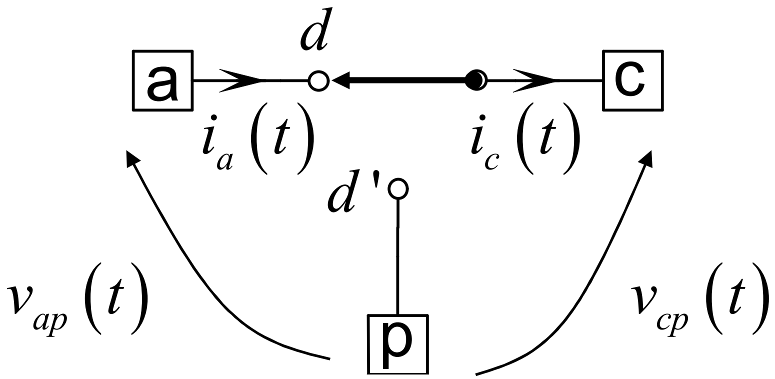
- Some functions require mathematical abstraction: duty ratio



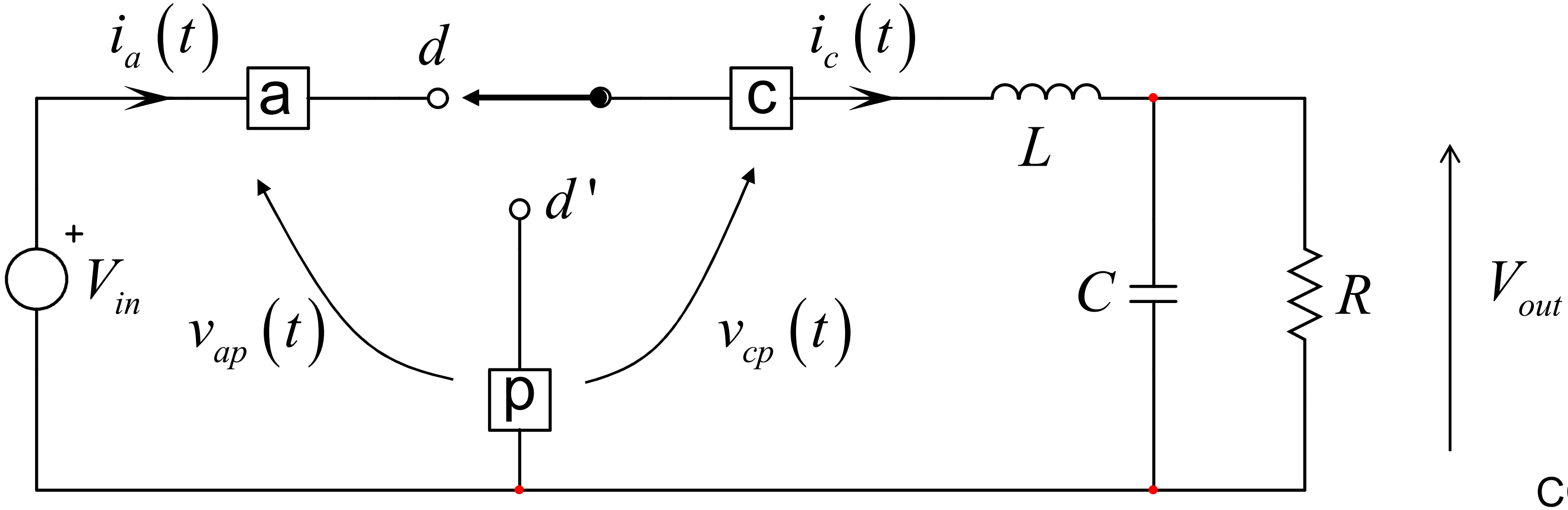
- At the modulation frequency scale, points look contiguous
- Link them through a continuous-time ripple-free function $d(t)$

The Common Passive Configuration

- The PWM switch is a single-pole double-throw model



- Install it in a buck converter and draw the waveforms



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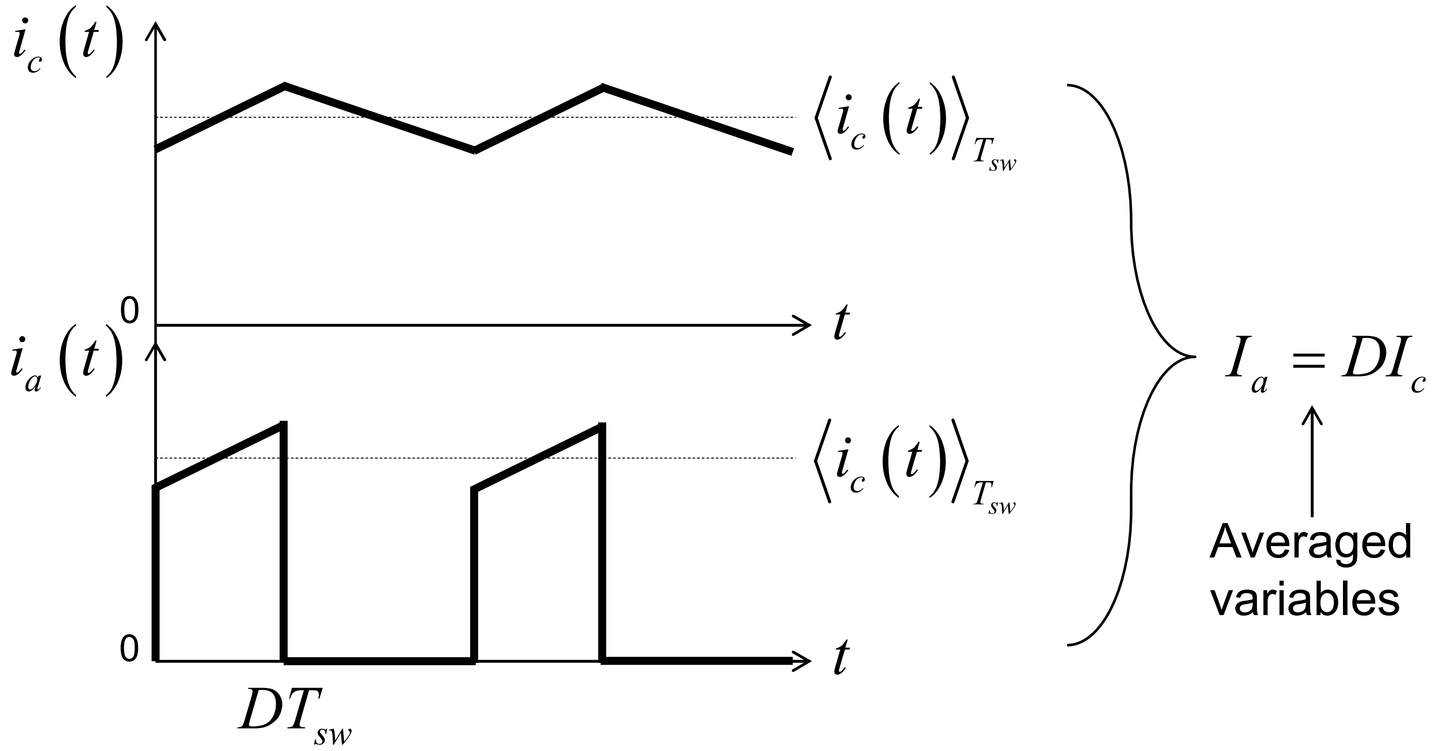
The Common Passive Configuration

- Average the current waveforms across the PWM switch



$$\langle i_a(t)\rangle_{T_{sw}} = I_a = \frac{1}{T_{sw}}\int_0^{dT_{sw}} i_a(t)\,dt = D\langle i_c(t)\rangle_{T_{sw}} = DI_c$$

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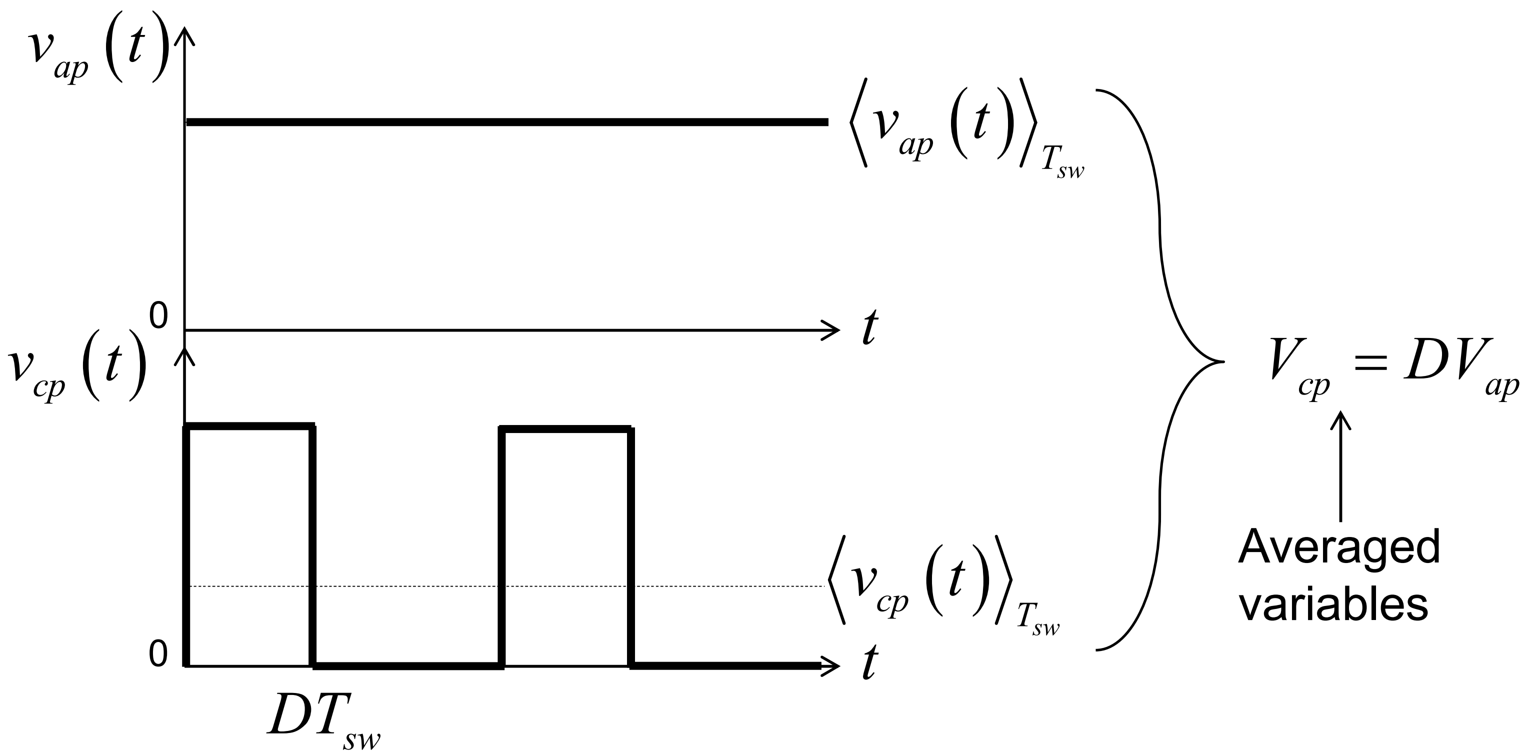
The Common Passive Configuration

❑ Average the voltage waveforms across the PWM switch



$$\langle v_{cp}(t)\rangle_{T_{sw}} = V_{cp} = \frac{1}{T_{sw}}\int_0^{dT_{sw}} v_{cp}(t)\,dt = D\langle v_{ap}(t)\rangle_{T_{sw}} = DV_{ap}$$

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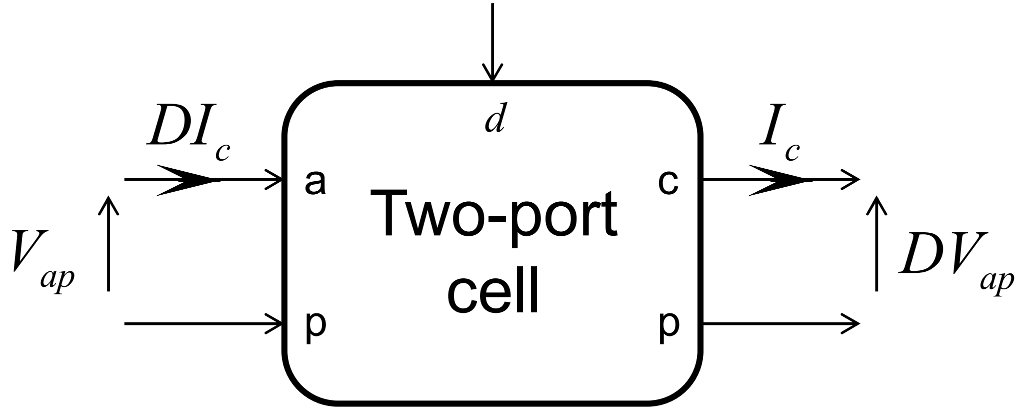
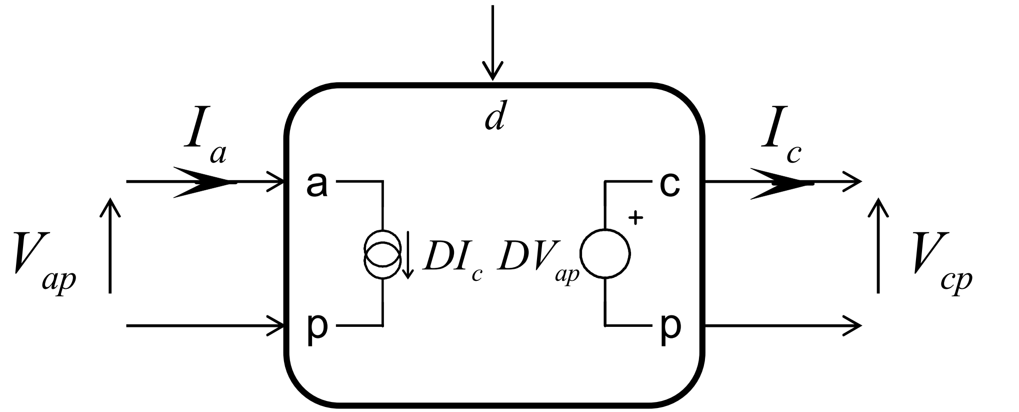
A Two-Port Representation

- We have a link between input and output variables



- It can further be illustrated with current and voltage sources



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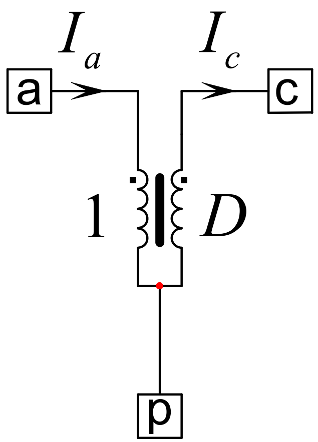
A Transformer Representation

- The PWM switch large-signal model is a dc "transformer"!

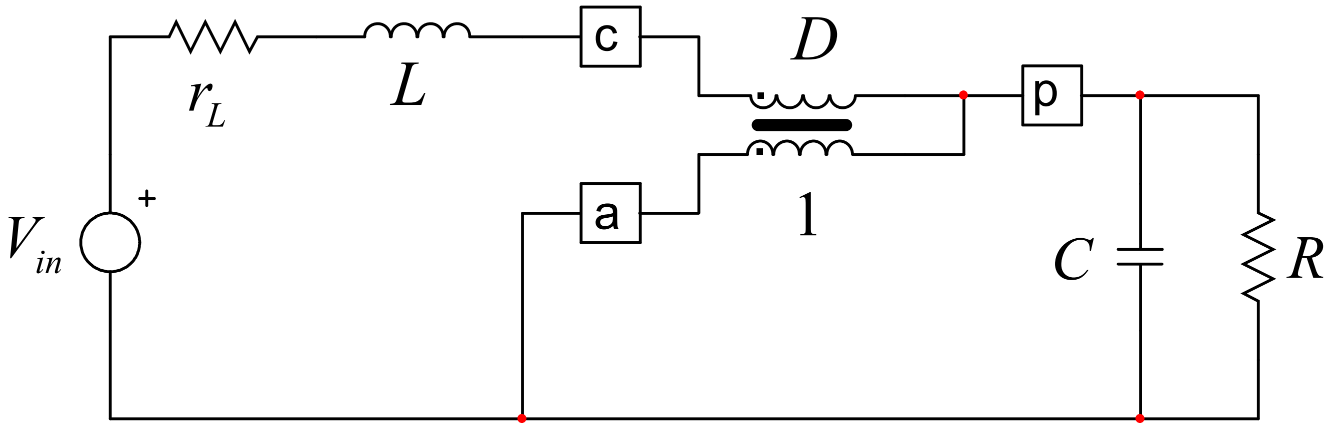


$$I_a = DI_c \qquad I_c = \frac{I_a}{D}$$

$$V_{ap} = \frac{V_{cp}}{D} \qquad V_{cp} = DV_{ap}$$

dc equations!

- It can be immediately plugged into any 2-switch converter



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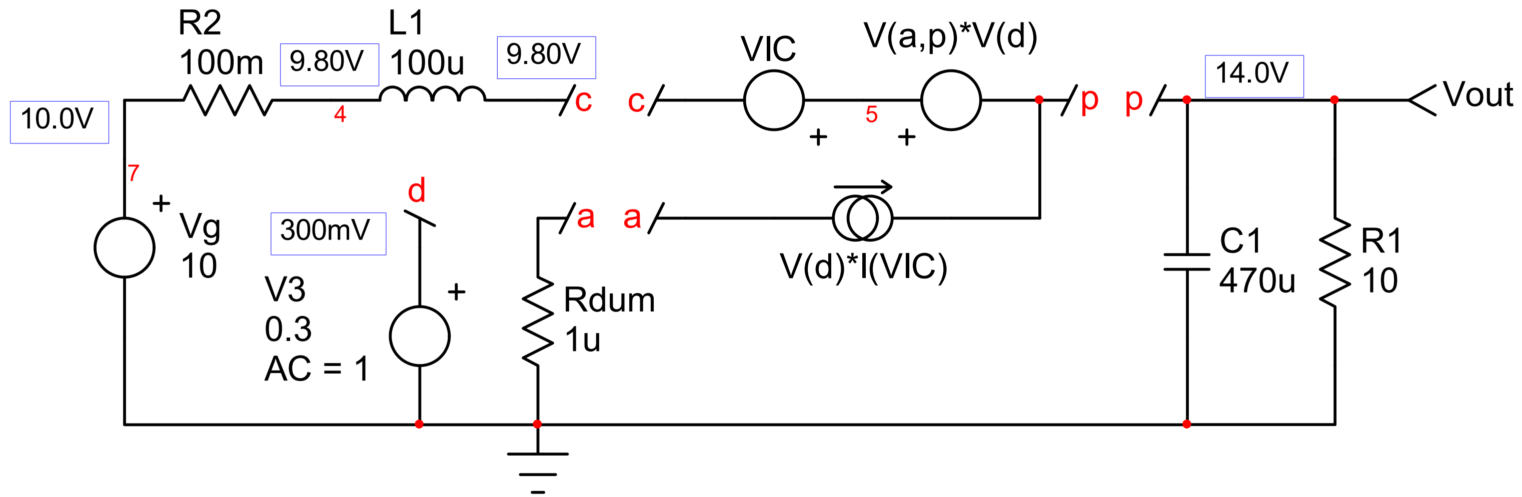
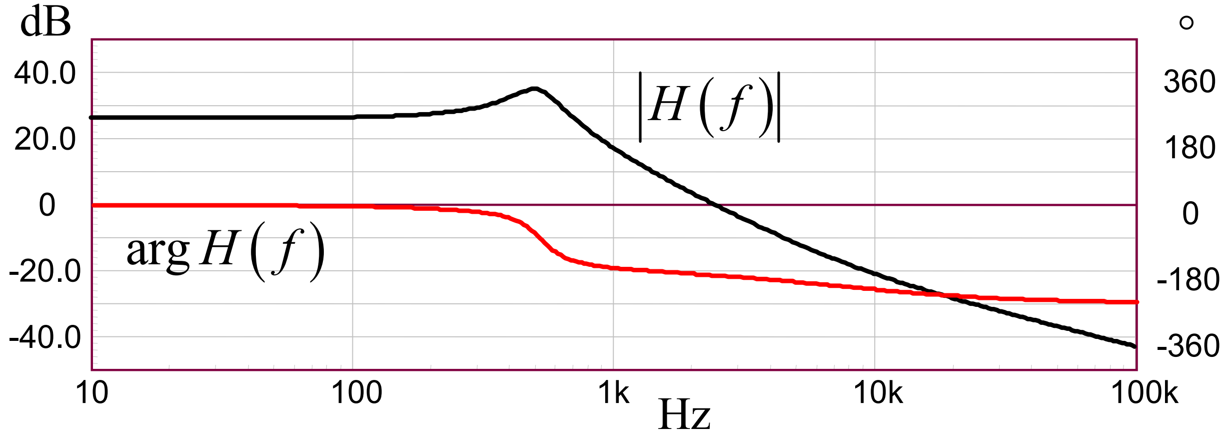
Simulate Immediately with this Model

- SPICE can get you the dc bias point



- …but also the ac response as it linearizes the circuit



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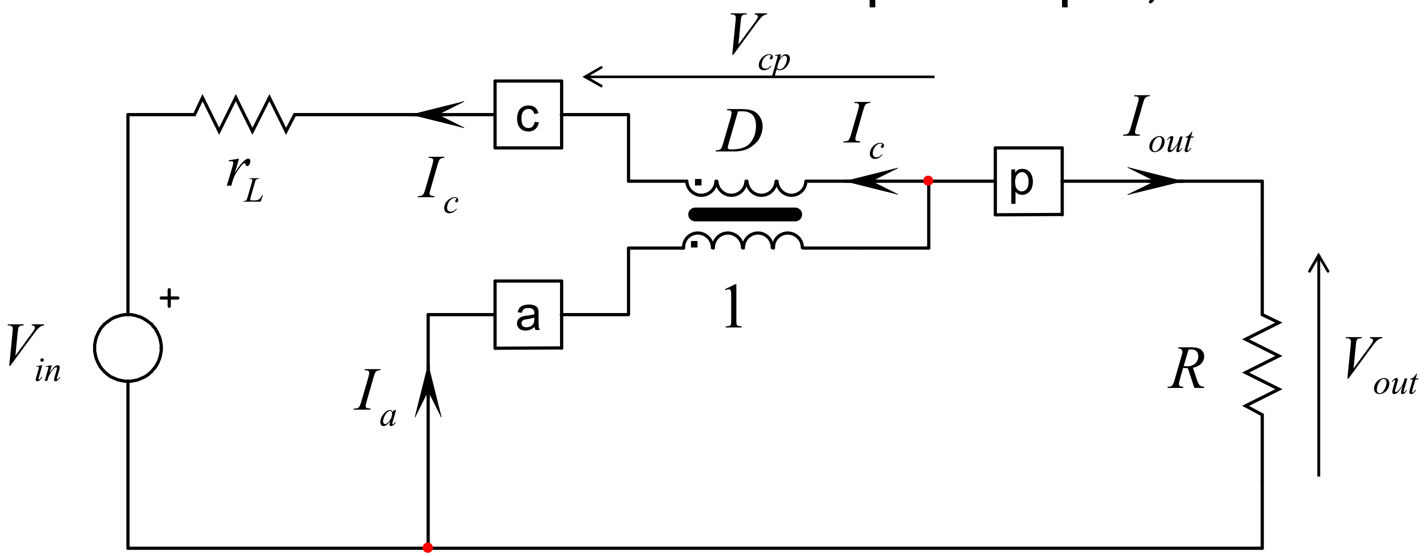
We Want Transfer Functions

- Derive the dc transfer function: open caps., short inductors



$$V_{out} = I_{out} R$$

$$V_{out} = \left(I_a - I_c\right) R$$

$$I_a = D I_c$$

$$V_{out} = I_c \left(D - 1\right) R$$

$$V_{in} + r_L I_c - V_{cp} = V_{out}$$

$$V_{in} + r_L I_c + D V_{out} = V_{out}$$

$$I_c = \frac{V_{out}\left(1 - D\right) - V_{in}}{r_L}$$

$$\frac{V_{out}}{V_{in}} = \frac{1}{\left(1 - D\right) - \dfrac{r_L}{\left(D - 1\right) R}}$$

$$\frac{V_{out}}{V_{in}} = \frac{1}{D'} \frac{1}{1 + \dfrac{r_L}{R D'^2}}$$

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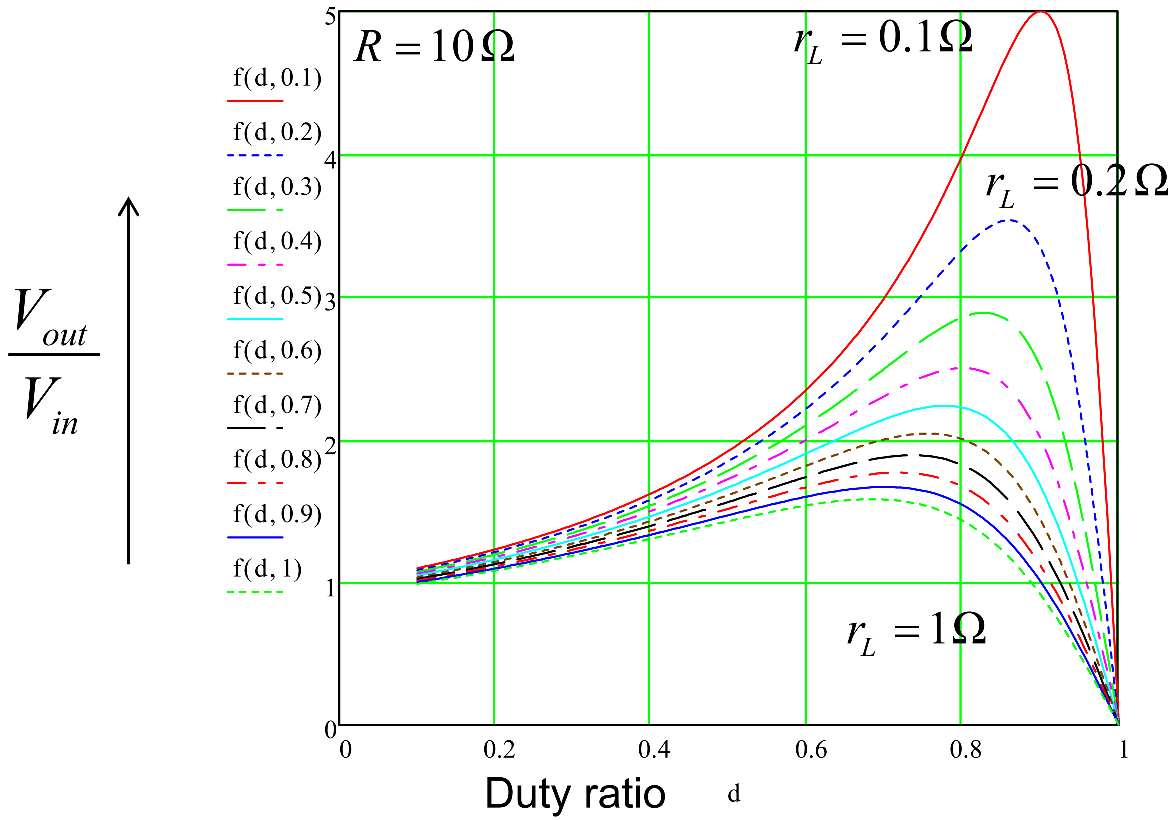
Plotting Transfer Functions

- Plot the lossy boost transfer function in a snapshot



- Above a certain conversion ratio, latch-up occurs

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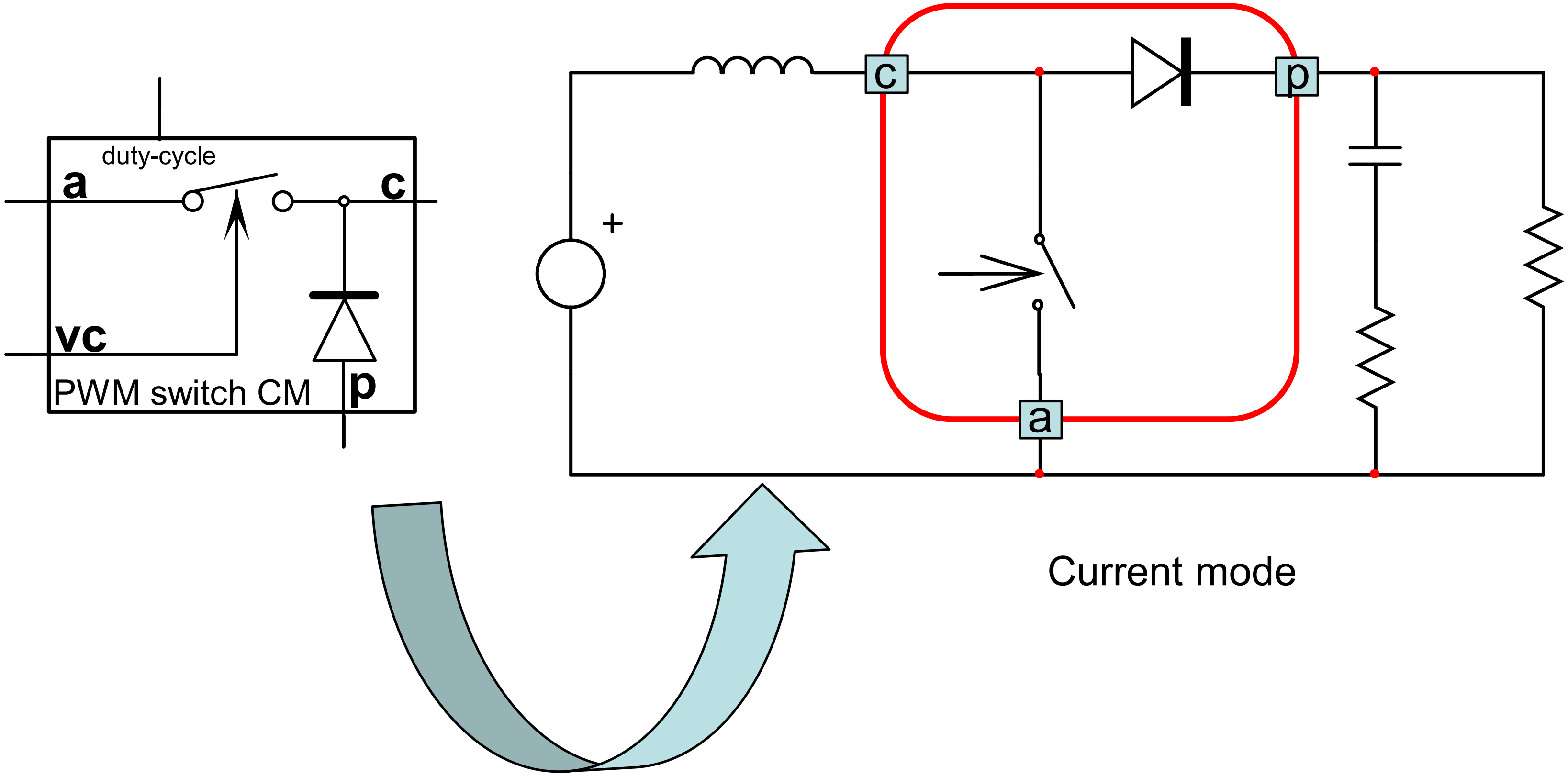
A Boost Converter in Current Mode

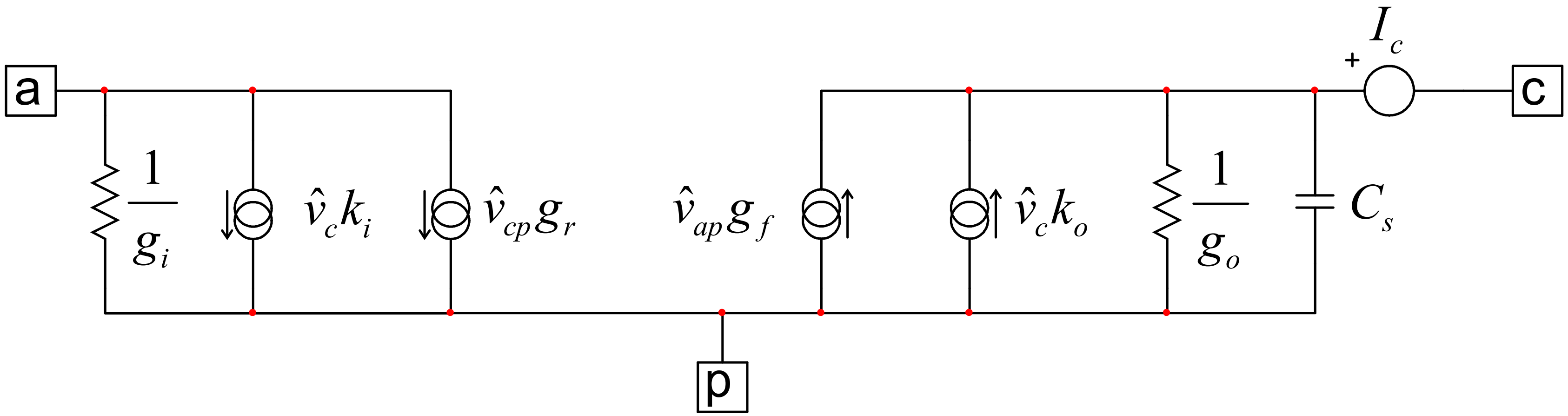
❑ Identify the diode and switch position in a boost CM



❑ Replace switches by the small-signal PWM switch model

A Small Signal Model

- The model includes current sources and conductances



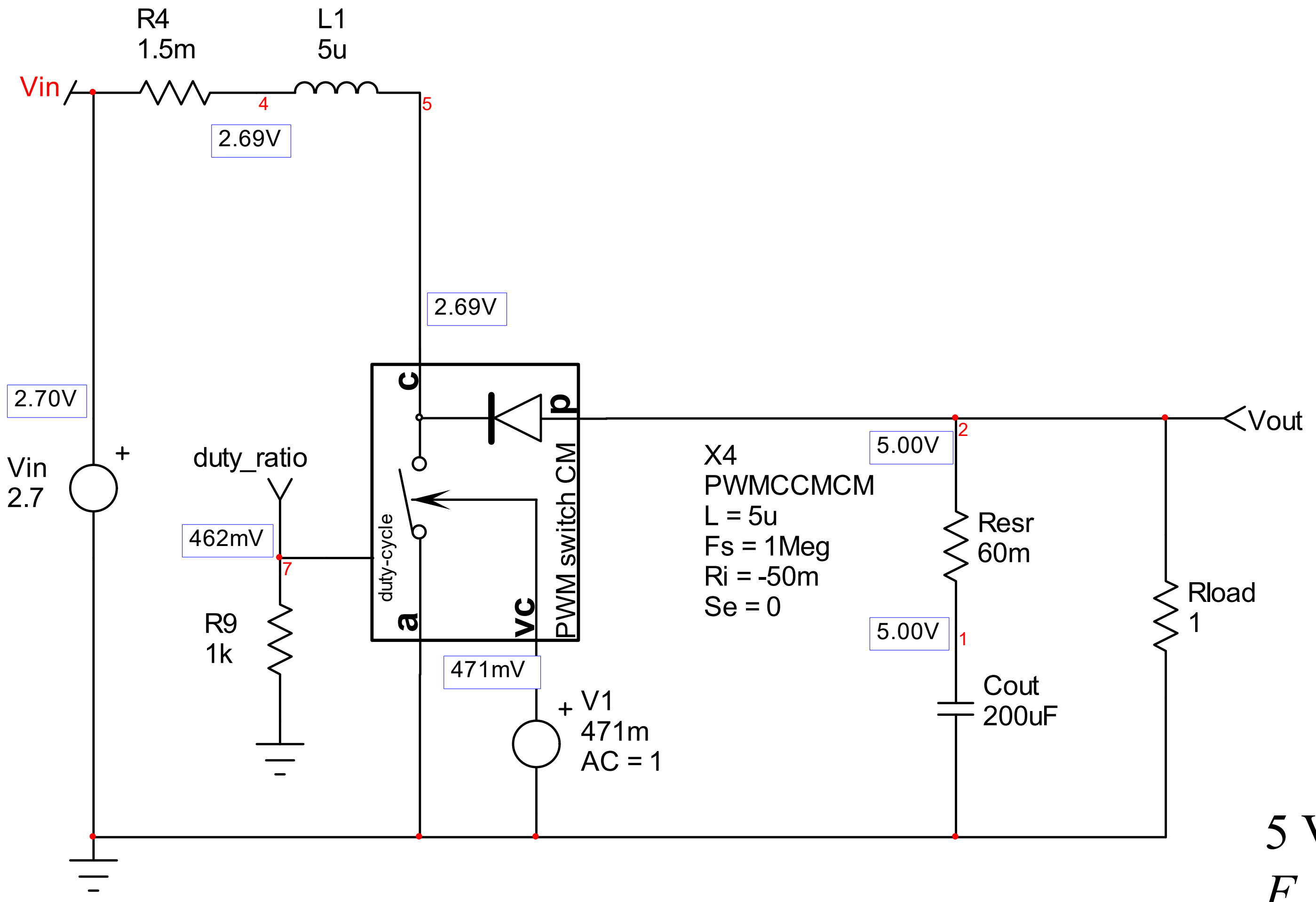
$$k_o = \frac{1}{R_i} \qquad g_f = Dg_o - \frac{DD'T_{sw}}{2L} \qquad g_o = -\frac{T_{sw}}{L}\left(D'\frac{S_a}{S_n} + \frac{1}{2} - D\right)$$

$$k_i = \frac{D}{R_i} \qquad g_i = D\left(g_f - \frac{I_c}{V_{ap}}\right) \qquad g_r = \frac{I_c}{V_{ap}} - g_o D$$

V. Vorpérian,"Analysis of Current-Controlled PWM Converters using the Model of the PWM Switch", PCIM Conference ,1990

Start with a Large Signal Response

- Use the original large-signal (nonlinear) PWMCM model

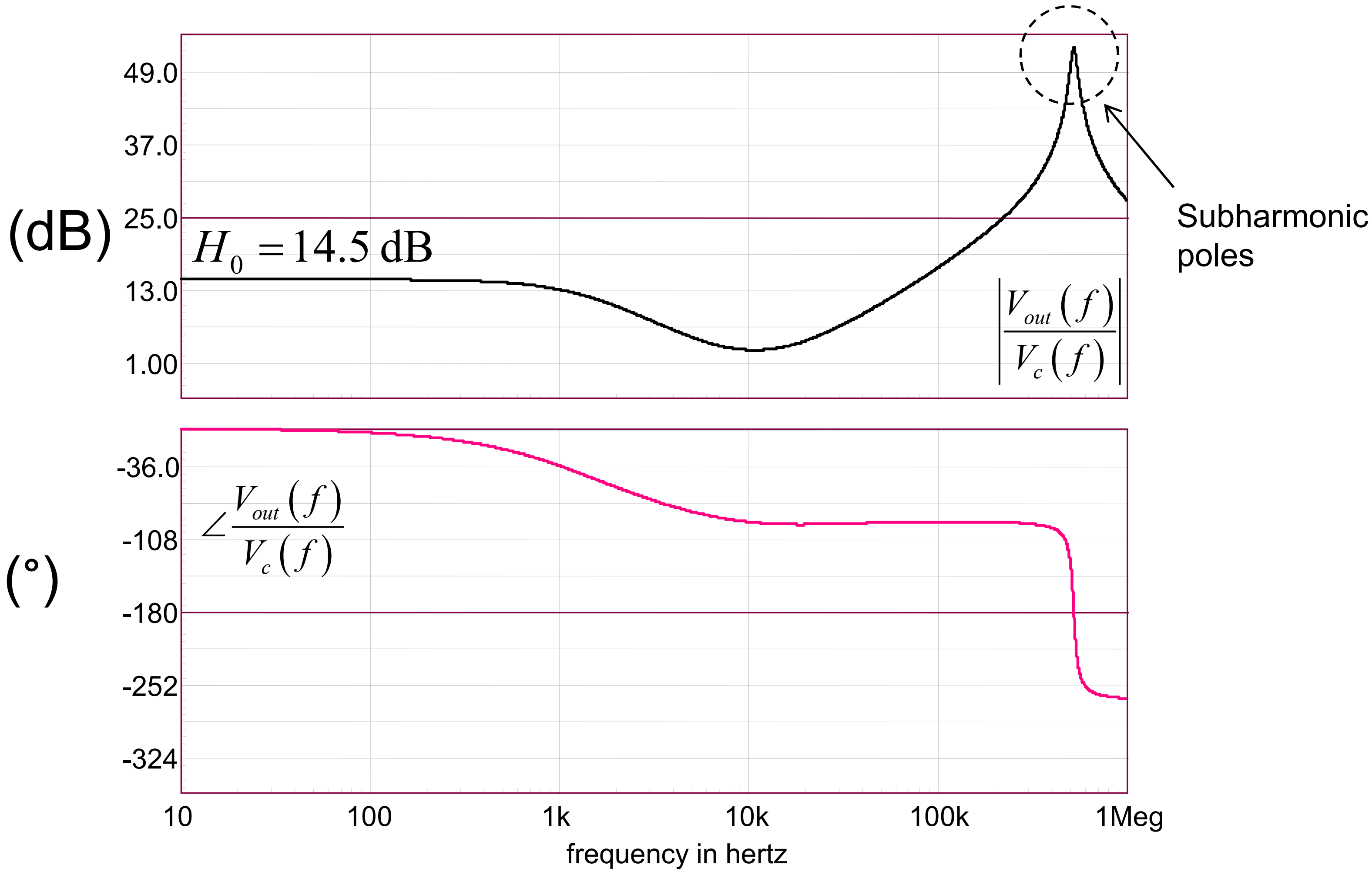


5 V/1 A

$F_{sw} = 1$ MHz

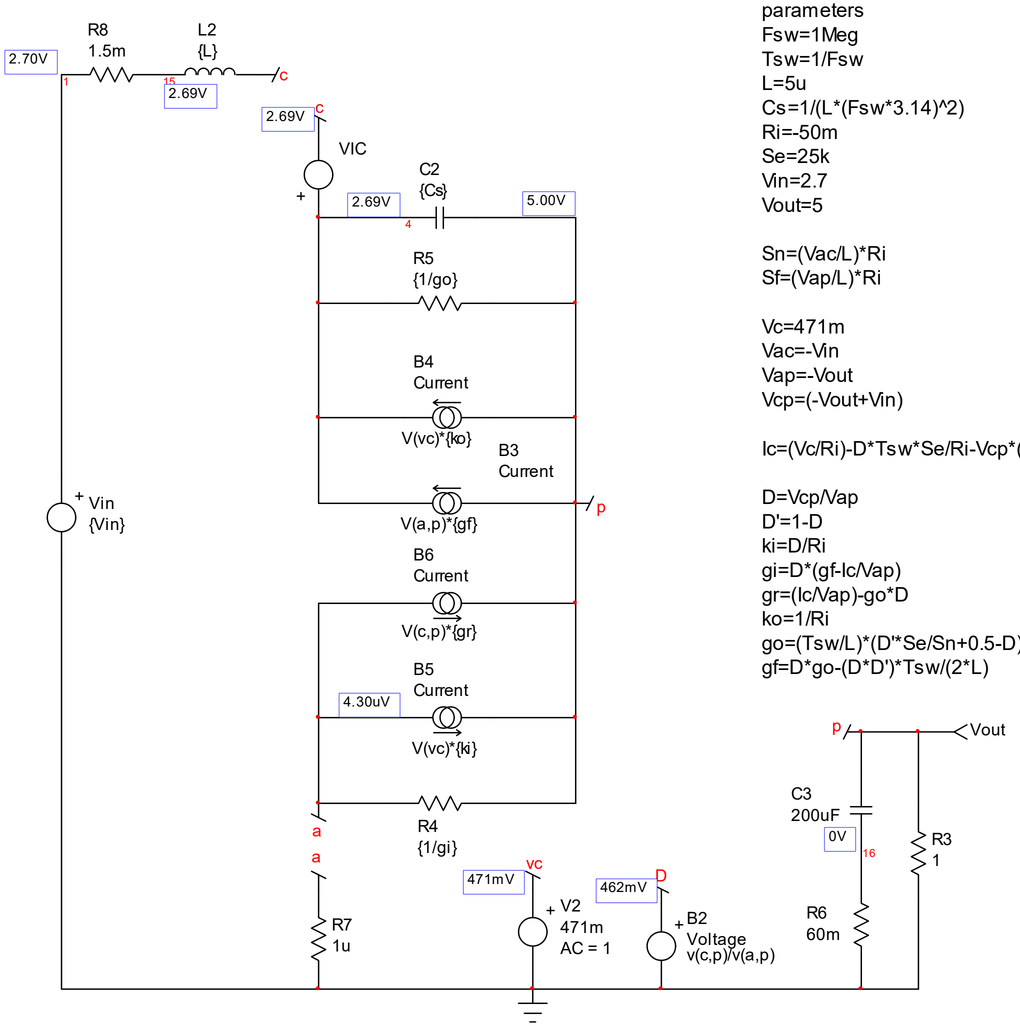
Frequency Response of the CM Boost

- This frequency response becomes the reference



Plug the Linearized Small-Signal Model

- Use the linearized model to check coefficients and response



parameters
Fsw=1Meg
Tsw=1/Fsw
L=5u
Cs=1/(L*(Fsw*3.14)^2)
Ri=-50m
Se=25k
Vin=2.7
Vout=5

Sn=(Vac/L)*Ri
Sf=(Vap/L)*Ri

Vc=471m
Vac=-Vin
Vap=-Vout
Vcp=(-Vout+Vin)

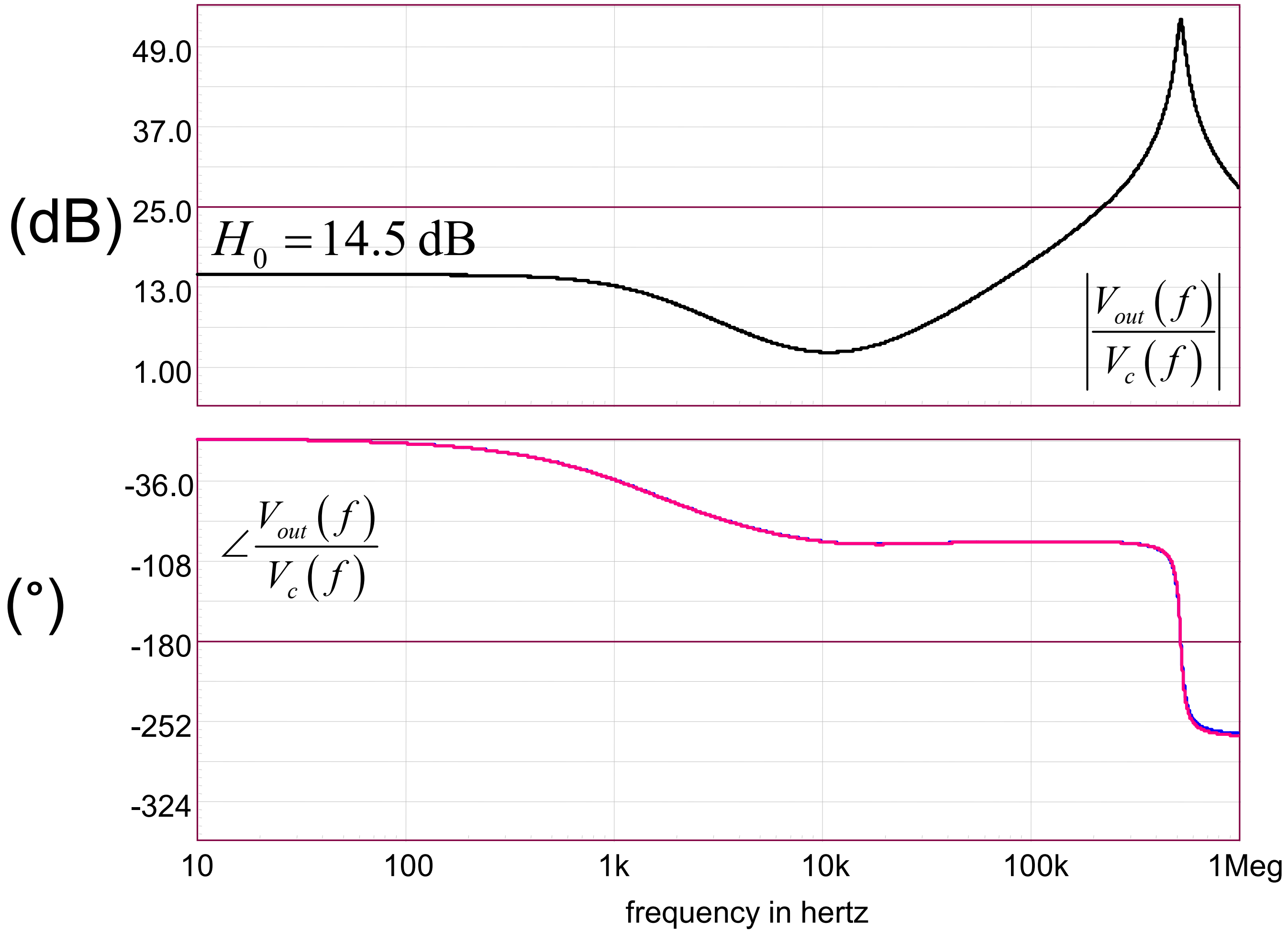
Ic=(Vc/Ri)-D*Tsw*Se/Ri-Vcp*(1-D)*Tsw/(2*L)

D=Vcp/Vap
D'=1-D
ki=D/Ri
gi=D*(gf-Ic/Vap)
gr=(Ic/Vap)-go*D
ko=1/Ri
go=(Tsw/L)*(D'*Se/Sn+0.5-D)
gf=D*go-(D*D')*Tsw/(2*L)

These coefficients are computed by the macro

Small-Signal Response and Original Model

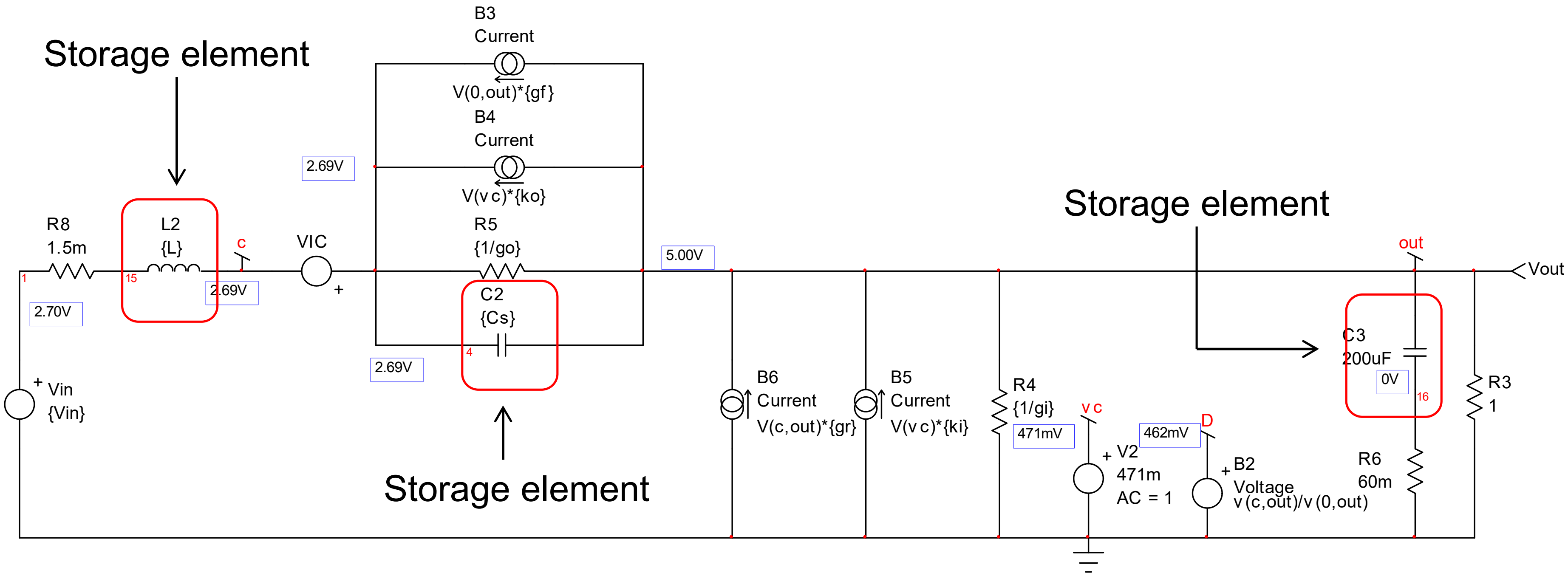
- ❑ Validate the small-signal approach by comparing responses



Good to go!

Rearrange Sources to Improve Readability

- ❑ It is important to lay the network out properly



- ❑ Make sure the frequency response remains unchanged

What is the Converter Order?

- Count storage elements: L and two C
- The denominator of the transfer function is of 3rd order

Dc gain

Zeros

$$H(s) = H_0 \frac{\left(1+\frac{s}{s_{z_1}}\right)\left(1+\frac{s}{s_{z_2}}\right)}{1+\frac{s}{s_{p_1}}} \frac{1}{1+\frac{s}{\omega_0 Q}+\left(\frac{s}{\omega_0}\right)^2}$$

Subharmonic poles

Start with the Static Analysis

- For $s = 0$, short inductors and open capacitors

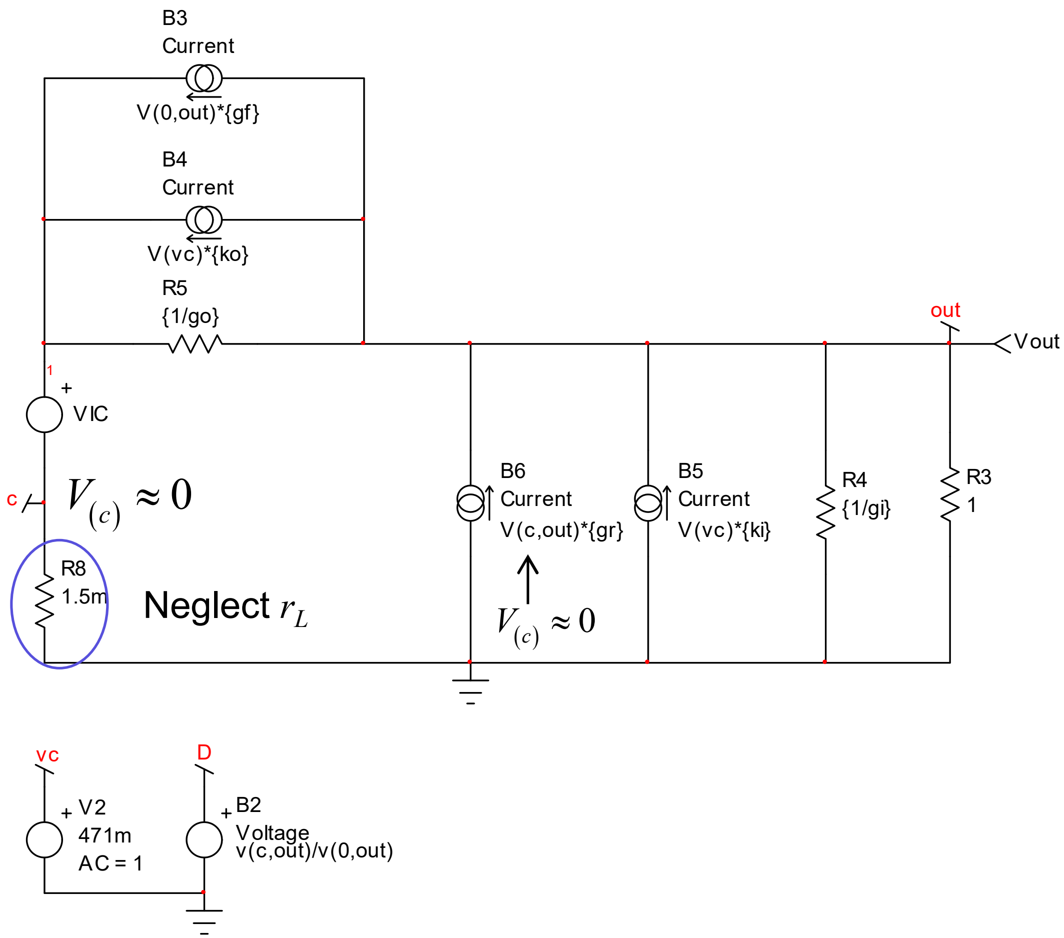
parameters
Fsw=1Meg
Tsw=1/Fsw
L=5u
Cs=1/(L*(Fsw*3.14)^2)
Ri=-50m
Se=0
Vin=2.7
Vout=5

Sn=(Vac/L)*Ri
Sf=(Vap/L)*Ri

Vc=471m
Vac=-Vin
Vap=-Vout
Vcp=(-Vout+Vin)

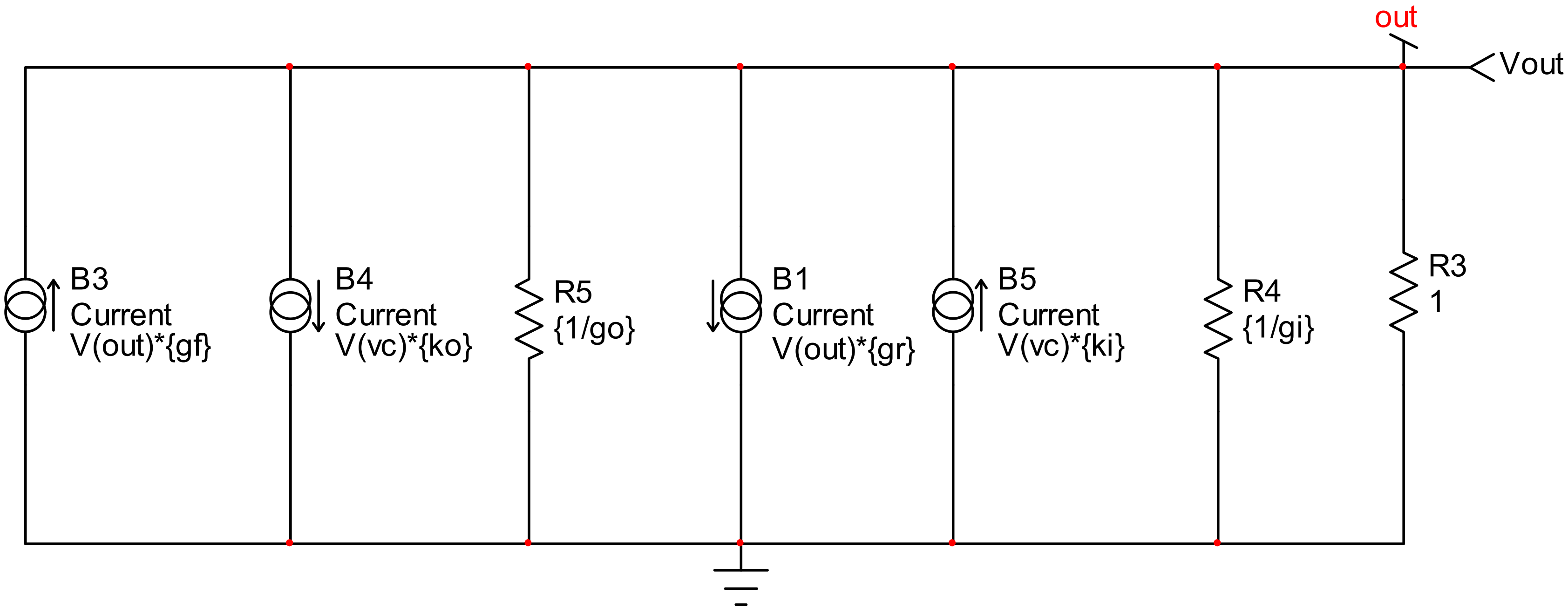
Ic=(Vc/Ri)-D*Tsw*Se-Vcp*(1-D)*Tsw/(2*L)

D=Vcp/Vap
D'=1-D
ki=D/Ri
gi=D*(gf-Ic/Vap)
gr=(Ic/Vap)-go*D
ko=1/Ri
go=(Tsw/L)*(D'*Se/Sn+0.5-D)
gf=D*go-(D*D')*Tsw/(2*L)



Simplify and Rearrange

- ❑ Rearrange sources to make the circuit look simpler



$$V_{out} g_f - V_c k_o - V_{out}\left(g_o + g_i\right) - V_{out} g_r + V_c k_i = \frac{V_{out}}{R_1}$$

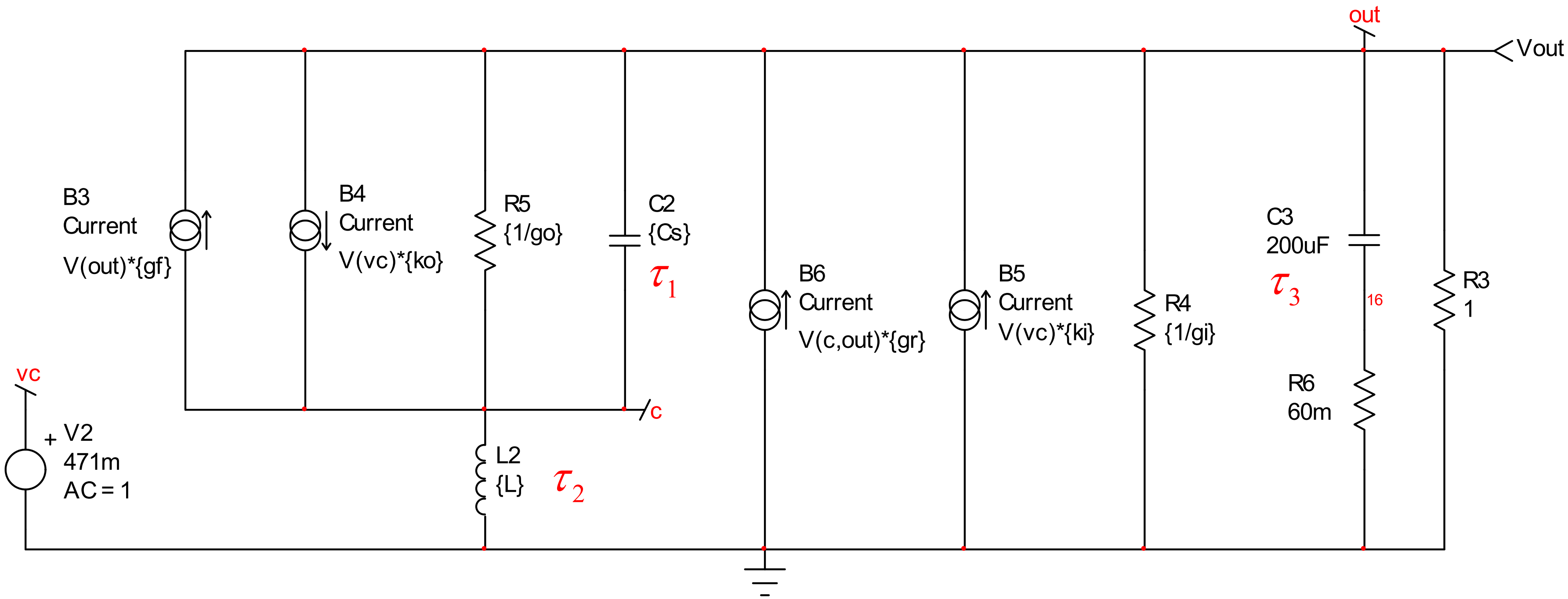
$$\Rightarrow \quad H_0 = \frac{k_0 - k_i}{g_f - \left(g_0 + g_i\right) - g_r - \dfrac{1}{R_1}}$$

Mathcad® calculations

$$H_0 := 20 \cdot \log\left[\frac{k_o - k_i}{g_f - \left(g_o + g_i\right) - g_r - \dfrac{1}{R_L}}\right] = 14.563 \text{ dB}$$

Put Storage Elements Back in Place

- ❑ Rearrange sources to make the circuit look simpler



- ❑ Label each storage element with a time constant τ
- ❑ Time to call the FACTs! Set excitation to 0, $V_c = 0$

FACTS: Fast Analytical Circuit TechniqueS

Polynomial Form of a 3rd Function

- A transfer function is made of a numerator and a denominator

$$H(s) = \frac{N(s)}{D(s)} \quad \begin{matrix} \longleftarrow \text{zeros} \\ \longleftarrow \text{poles} \end{matrix}$$

- The denominator combines the circuit time constants

$$D(s) = 1 + (\tau_1 + \tau_2 + \tau_3)s + \left(\tau_1\tau_2^1 + \tau_1\tau_3^1 + \tau_2\tau_3^2\right)s^2 + \left(\tau_1\tau_2^1\tau_3^{12}\right)s^3$$

Time constant with C_s (τ_1)

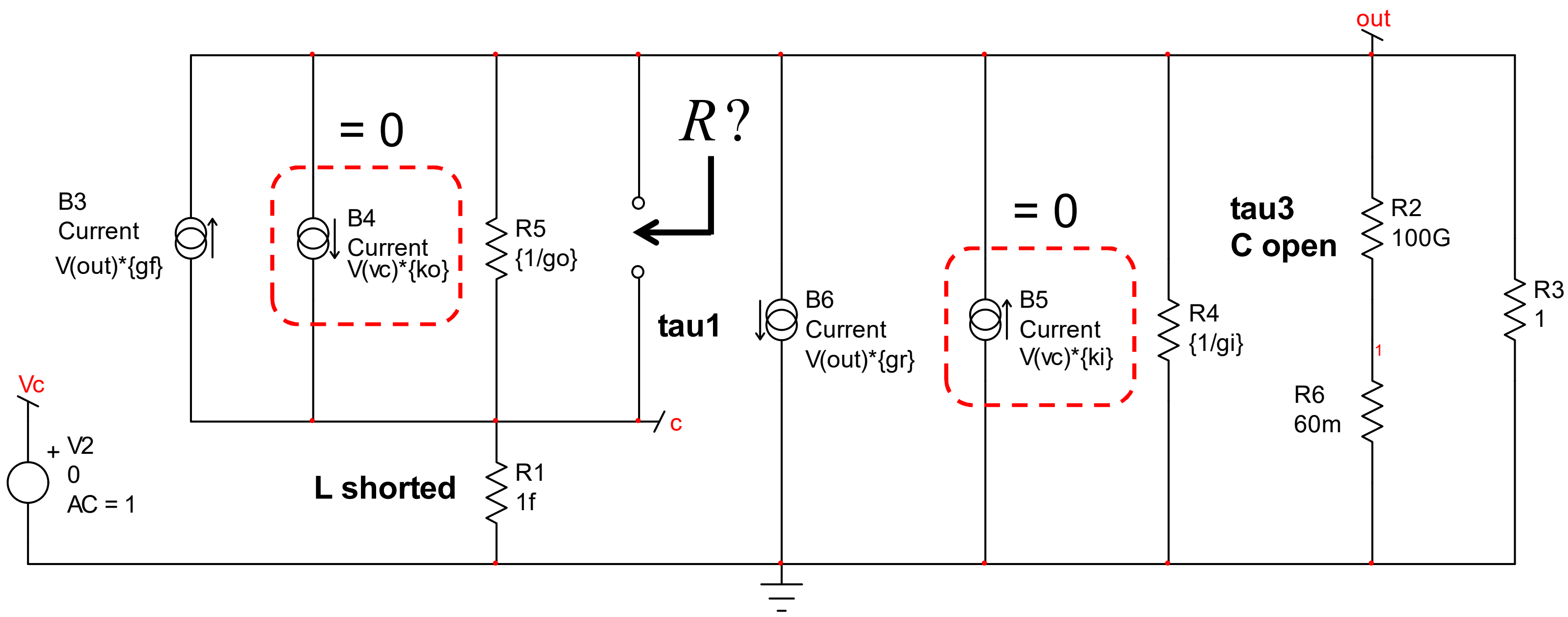
Time constant with L (τ_2)

Time constant with C_{out} (τ_3)

- To calculate time constants, suppress the excitation, all is dc

Calculate the First Time Constant

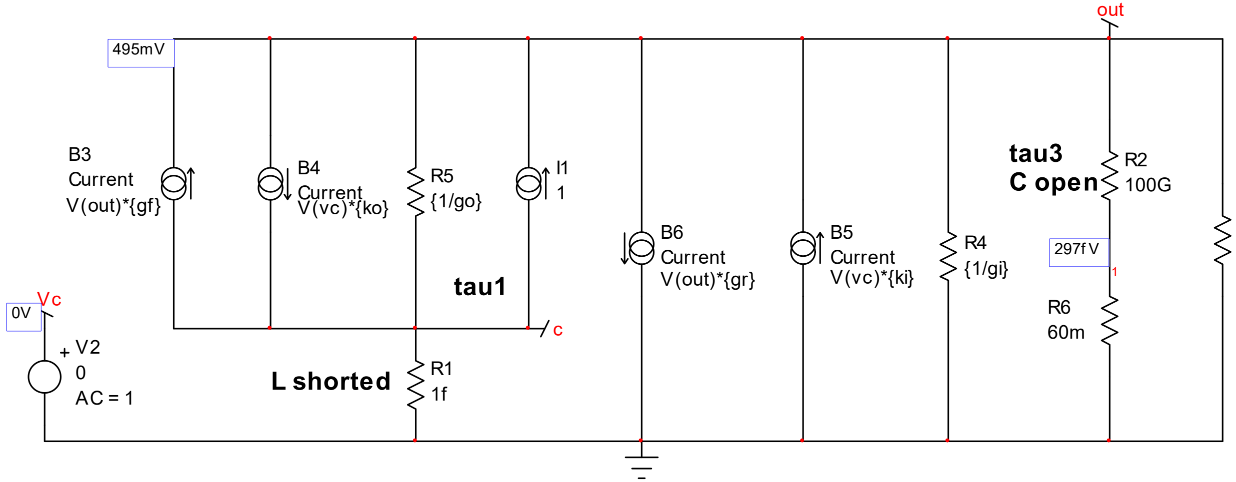
- ❑ What resistance does C_s « see » ?

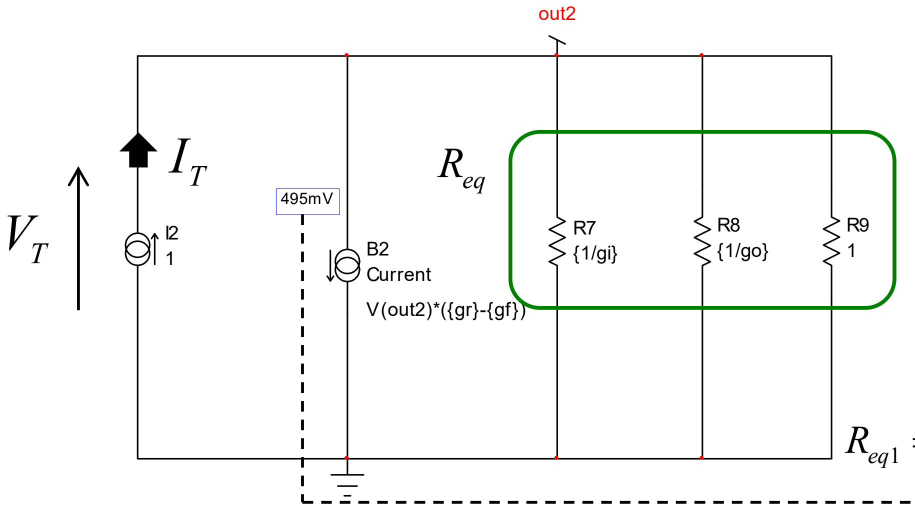


- ❑ The excitation is 0 so several sources go away
- ❑ You can further simplify the circuit and test it in SPICE

Calculate the First Time Constant

- Check the result with a simple dc operating point calculation





$$I_T = V_{out}\left(g_r - g_f\right) + \frac{V_{out}}{R_{eq}}$$

$$R = \frac{V_T}{I_T} = \frac{V_{out}}{I_T} = \frac{1}{g_r - g_f + \frac{1}{R_{eq}}}$$

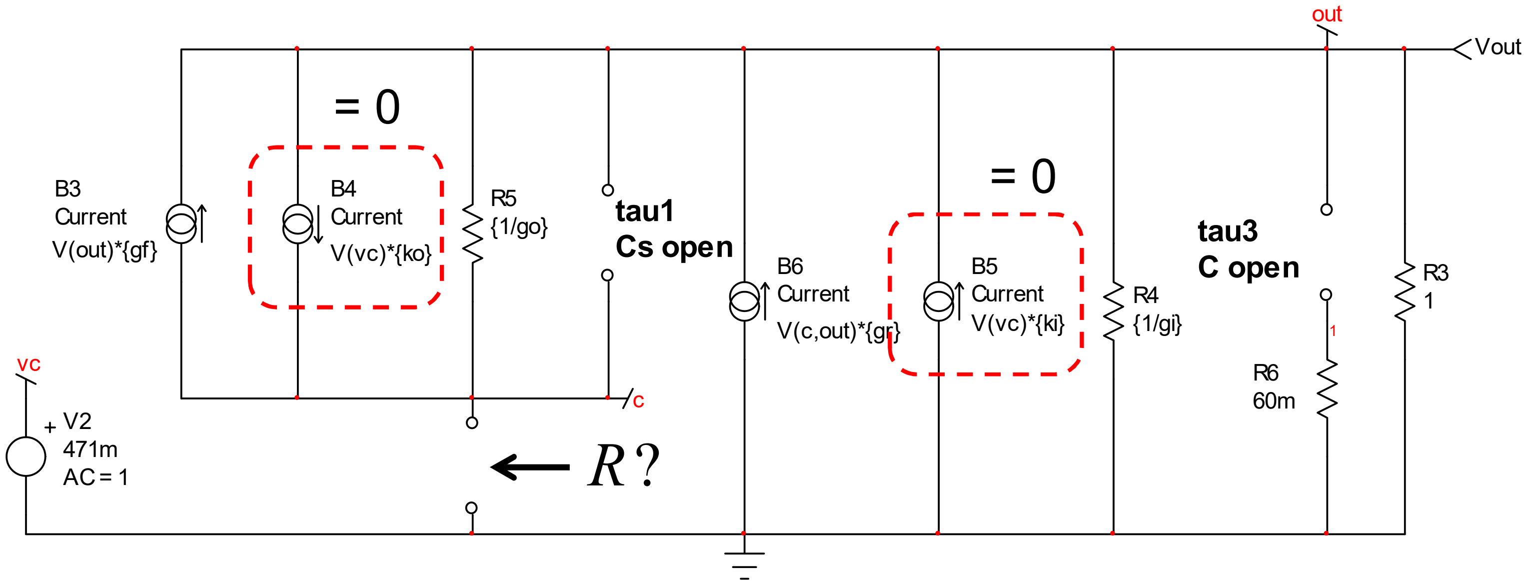
$$R_{eq1} = R_L \,||\, \frac{1}{g_i} \,||\, \frac{1}{g_o}$$

$$\frac{1}{\frac{1}{R_{eq}} + g_r - g_f} = 0.495\Omega$$

$$\tau_1 = RC_s$$

Calculate the Second Time Constant

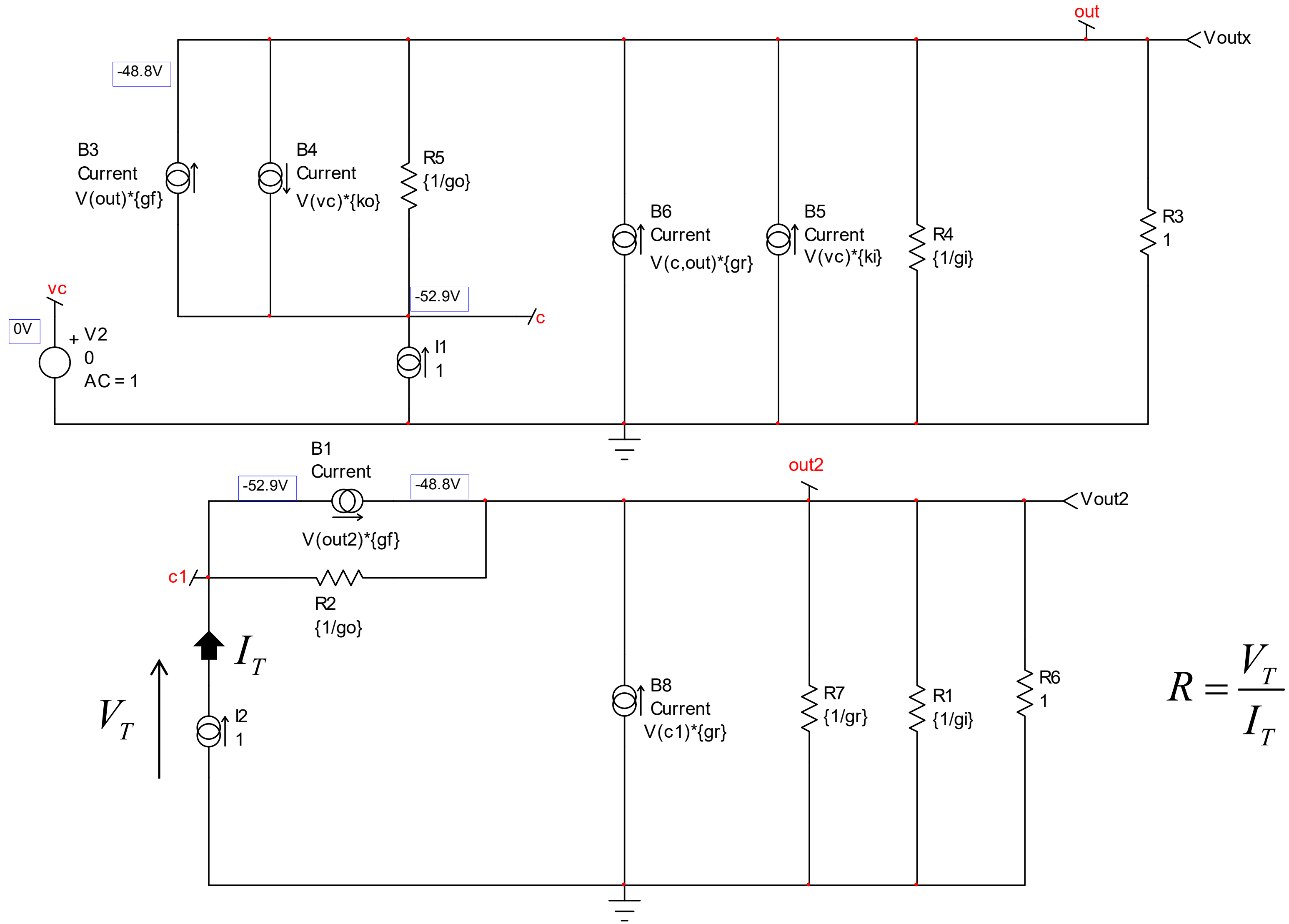
❑ What resistance does L « see » ?



❑ The excitation V_c is 0 so simplify the circuit

❑ Rearrange the network and test it in SPICE

Calculate the Second Time Constant



$$R = \frac{V_T}{I_T}$$

Calculate the Second Time Constant

- Express I_T and V_{out} then rearrange to unveil V_T/I_T

$$I_T = V_{out} g_f + \frac{V_T - V_{out}}{\frac{1}{g_0}} \qquad V_{out} = \left(I_T + V_T g_r\right) R_{eq} \qquad R_{eq2} = R_L \,||\, \frac{1}{g_i} \,||\, \frac{1}{g_r}$$

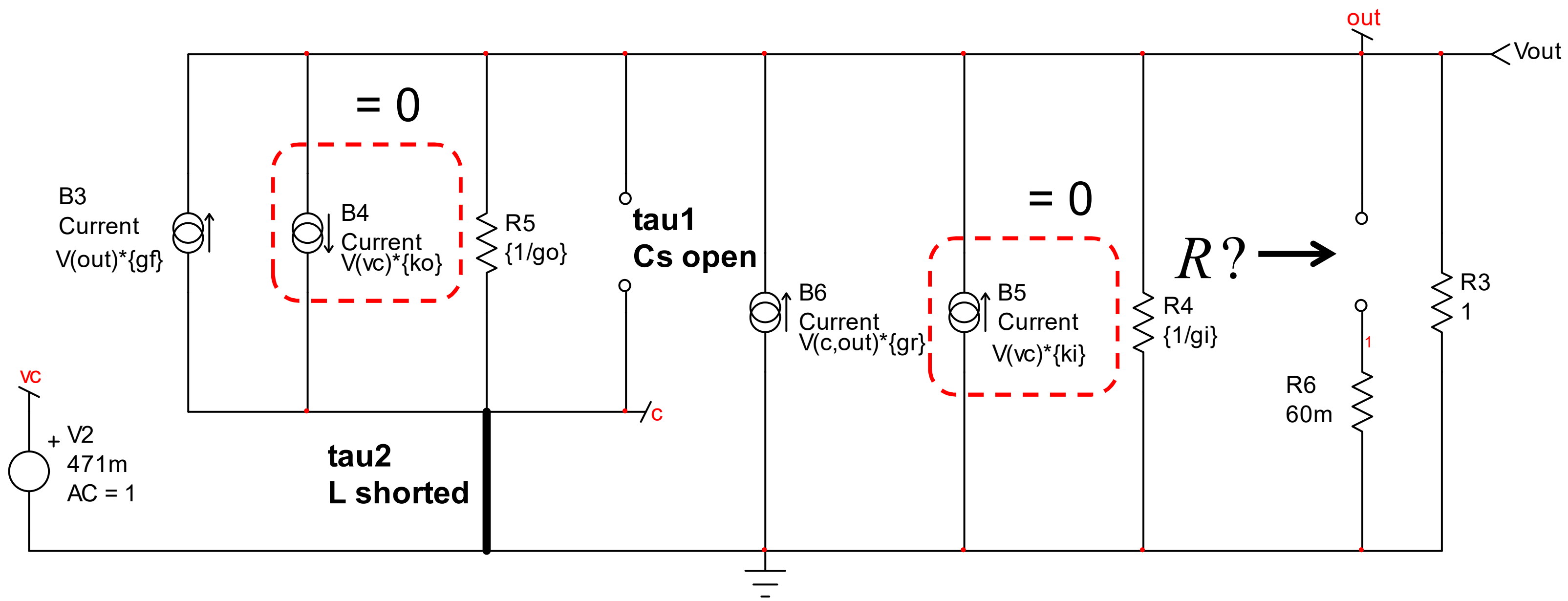
$$\Rightarrow \quad R = \frac{V_T}{I_T} = \frac{1 - R_{eq}\left(g_f - g_o\right)}{g_o + R_{eq} g_r \left(g_f - g_o\right)} \quad \Rightarrow \quad \tau_2 = \frac{L}{\frac{1 - R_{eq}\left(g_f - g_o\right)}{g_o + R_{eq} g_r \left(g_f - g_o\right)}}$$

$$\frac{\left[1 - R_{eq2} \cdot \left(g_f - g_o\right)\right]}{\left[g_o + R_{eq2} \cdot g_r \cdot \left(g_f - g_o\right)\right]} = -52.897\Omega$$

Same result as dc point simulation

Calculate the Third Time Constant

❑ What resistance does C_{out} « see » ?

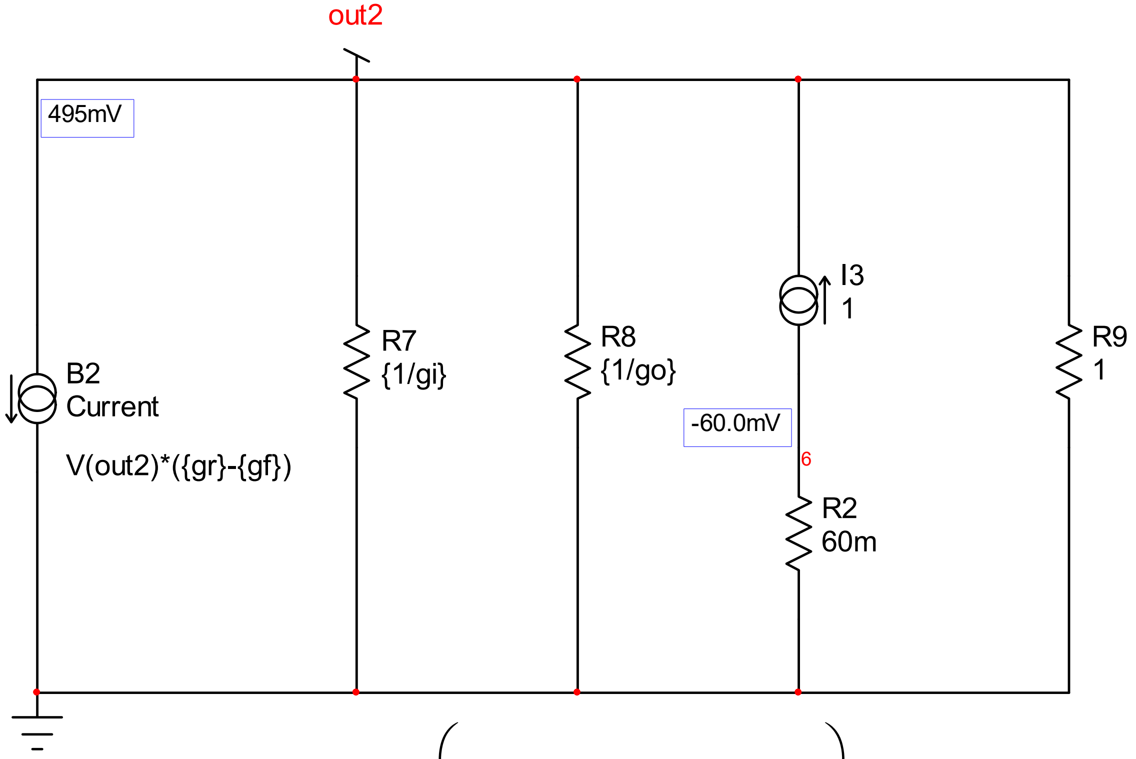


❑ The circuit is very close to that of τ_1

❑ Rearrange the network and test it in SPICE

Calculate the Third Time Constant

- Same resistance as for τ_1 plus r_C in series



$$\tau_3 = \left(\frac{1}{g_r - g_f + \dfrac{1}{R_{eq1}}} + r_C \right) C_{out}$$

$$R_{eq1} = R_L \,||\, \frac{1}{g_i} \,||\, \frac{1}{g_o}$$

First Coefficients a_1 and a_2

- FACTs tell us that a_1 sums up all time constants

$a_1 = \tau_1 + \tau_2 + \tau_3$ ⟶ Dimension is time

- For a_2, we multiply combined-time constants

$a_2 = \tau_1\tau_2^1 + \tau_1\tau_3^1 + \tau_2\tau_3^2$ ⟶ Dimension is time^2

- What is this new time constants definition, τ_2^1 ?

τ_2^1 | C_s (HF) / L C (dc) — ↑ R?

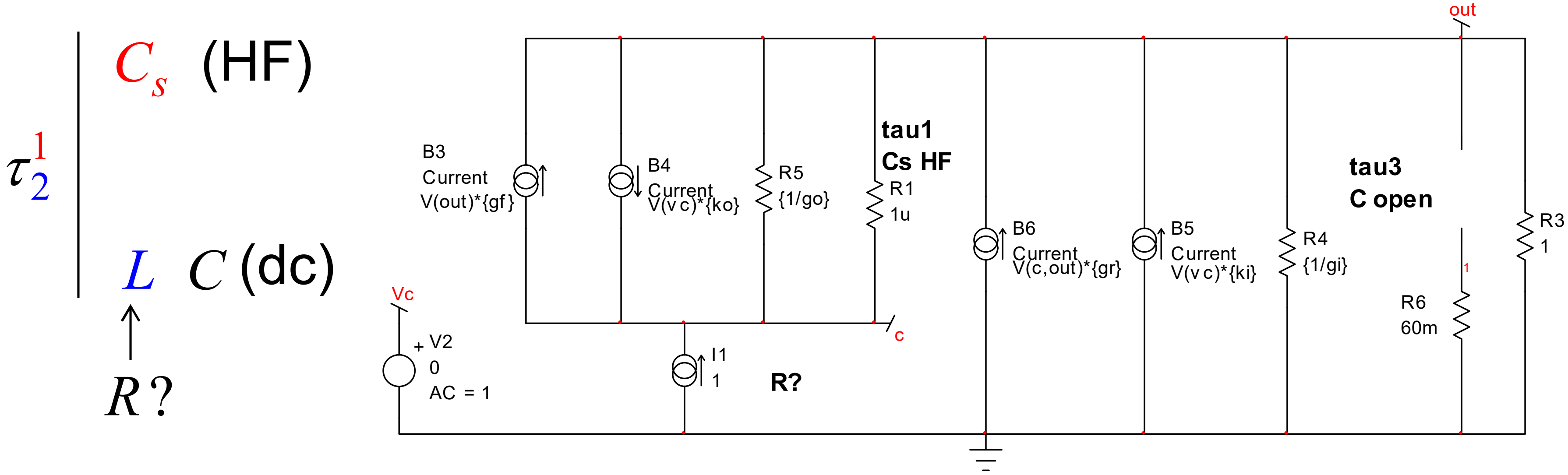
τ_3^1 | C_s (HF) / L (dc) C — ↑ R?

τ_3^2 | L (HF) / C C_s (dc) — ↑ R?

V. Vorpérian, "Fast Analytical Techniques for Electrical and Electronic Circuits", Cambridge Press, 2002

Calculate the Terms for a_2

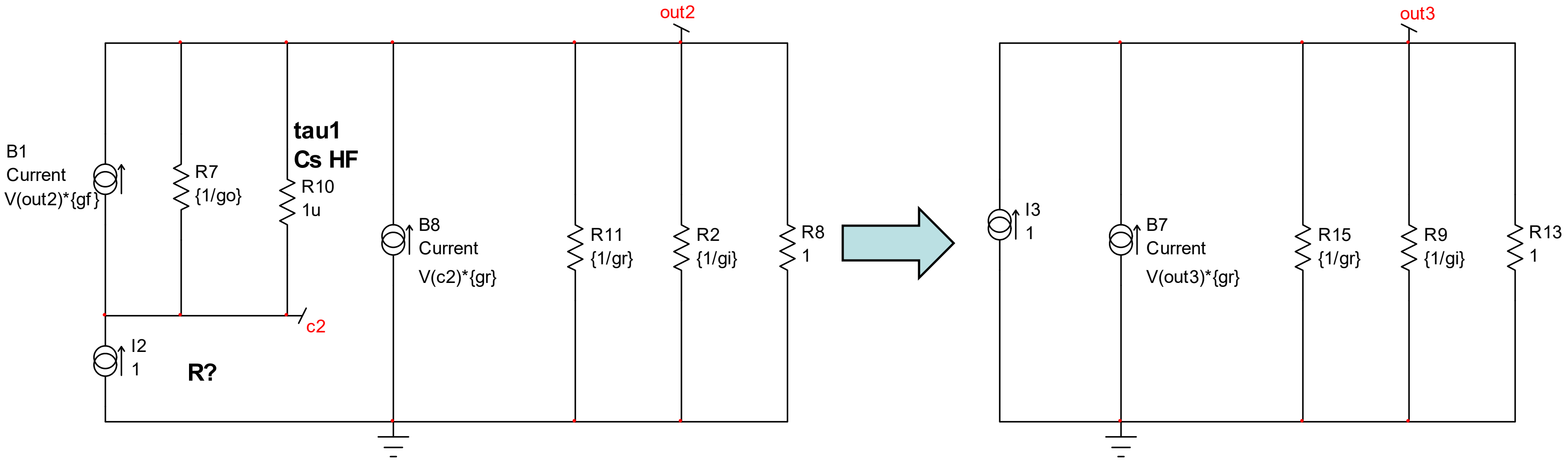
- What resistance does L while C_s is a short and C_{out} is open?



Go for an intermediate step

A Simplified Drawing for a Simple Answer

- What R? does L see while C_s is a short and C_{out} is open?



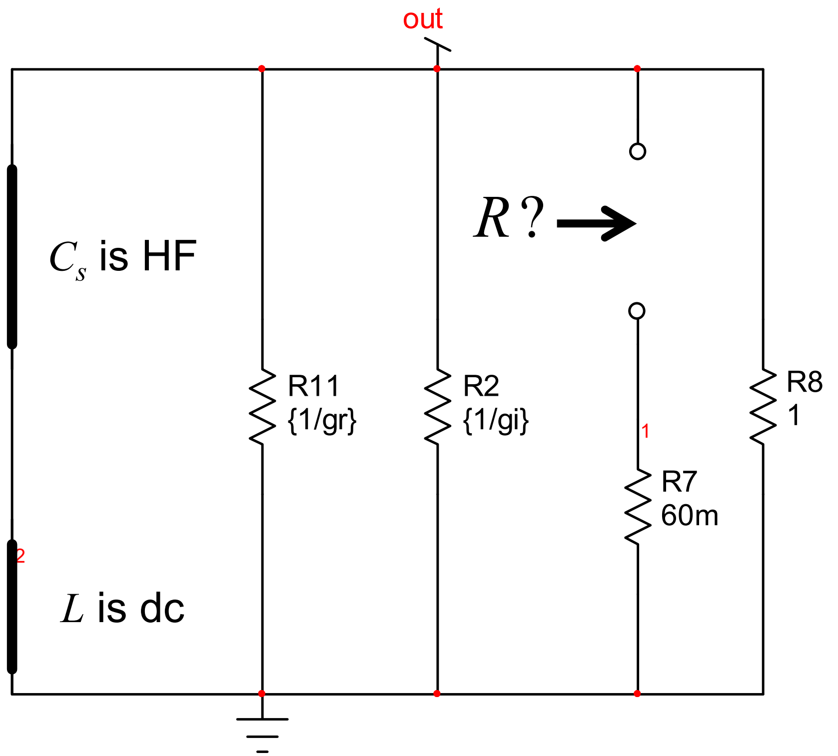
$$\tau_2^1 = \frac{L}{\dfrac{1}{\dfrac{1}{R_1 \,||\, \dfrac{1}{g_r} \,||\, \dfrac{1}{g_i}} - g_r}}$$

The Mid Term is Easy to Get

- ❑ L and C_s ensure a complete short over R_L

$$\tau_3^1 \begin{vmatrix} C_s \text{ (HF)} \\ \\ L \text{ (dc)} \quad C \end{vmatrix}$$

$\uparrow$ $R?$

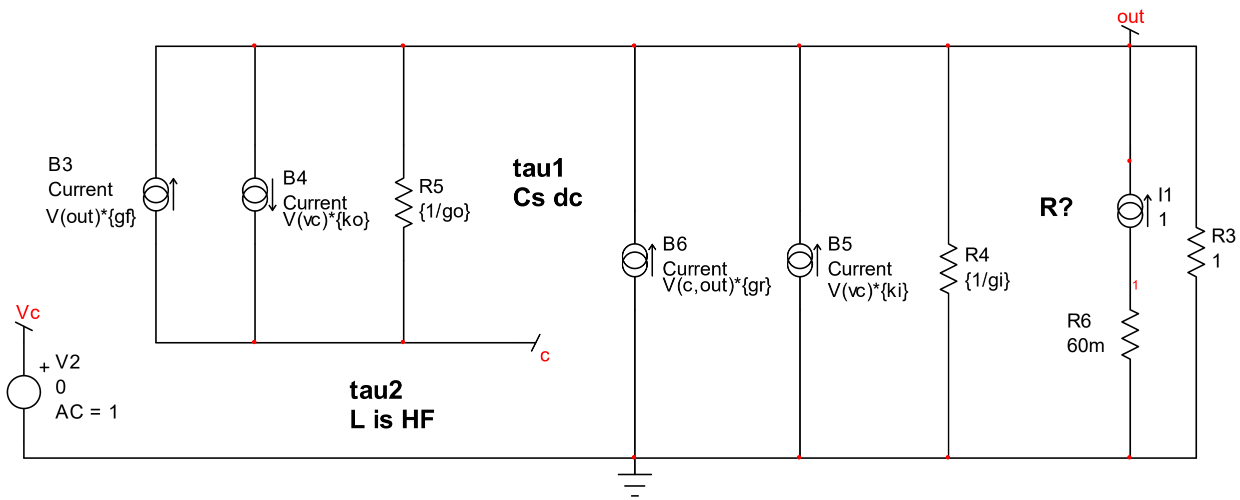


$$\tau_3^1 = r_C C_{out}$$

For the Last Term, L is open

- What resistance does C_{out} see when C_s and L are open?

τ_3^2 | L (HF)
C C_s (dc)
↑
R?

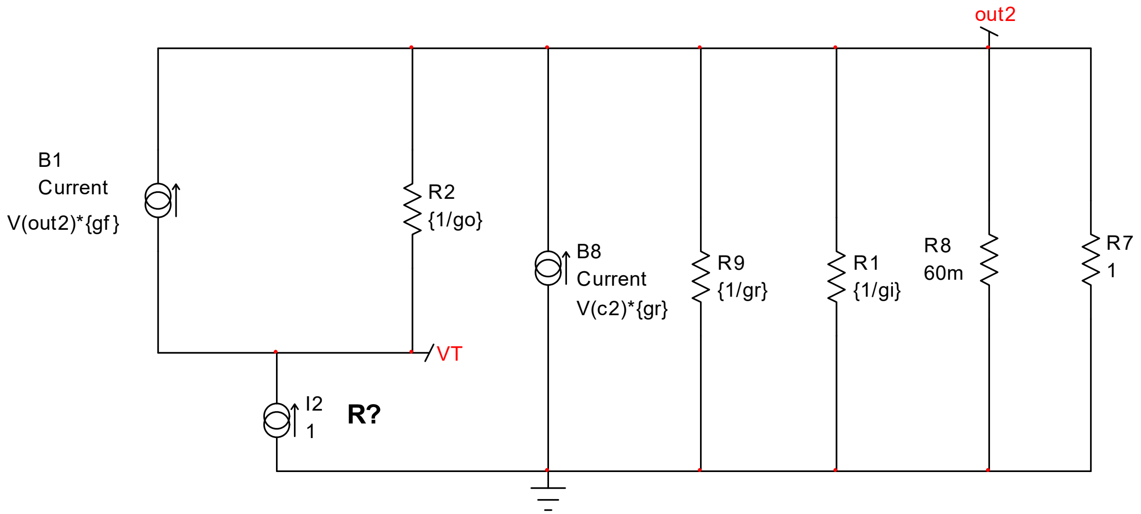


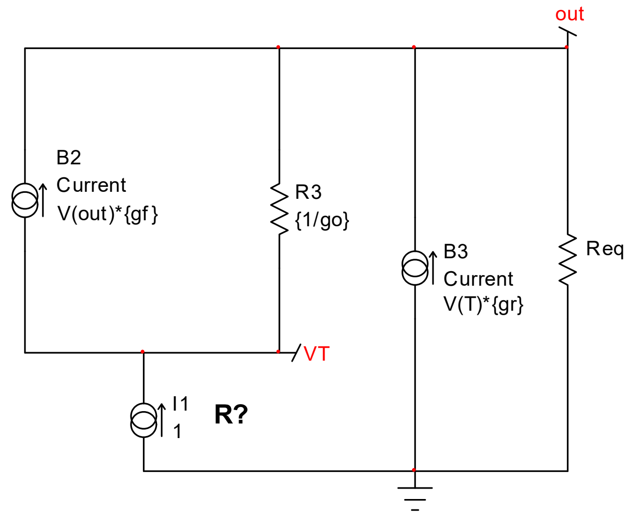
Too complex! Go for another combination.

$$\tau_2\tau_3^2 \rightarrow \tau_3\tau_2^3$$

Reorder Time Constants for Simpler Sketch

- ❑ The schematic now looks simpler…



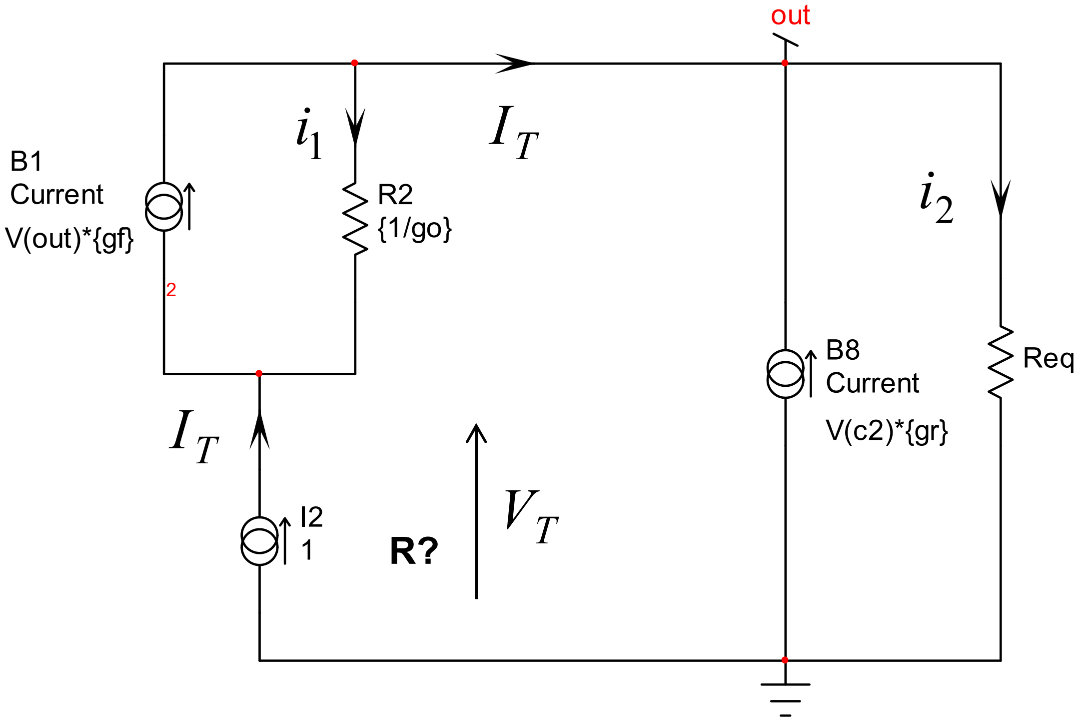


τ_2^3 | C_{out} (HF)

L C_s (dc)

↑ $R?$

Extract the New Time Constant Definition



$$R_{eq} = r_C \,||\, R_L \,||\, \frac{1}{g_i} \,||\, \frac{1}{g_r}$$

$$i_2 = I_T + V_T g_r$$

$$I_T = i_1 + V_{out} g_f$$

$$i_1 = \frac{V_T - V_{out}}{\frac{1}{g_o}}$$

$$V_{out} = R_{eq}\left(I_T + V_T g_r\right)$$

substitute

$$I_T = \frac{V_T - V_{out}}{\frac{1}{g_o}} + V_{out} g_f \;\Rightarrow\; R = \frac{1 - R_{eq}\left(g_f - g_o\right)}{g_0 + R_{eq} g_r \left(g_f - g_o\right)}$$

$$\tau_2^3 = \frac{L}{R}$$

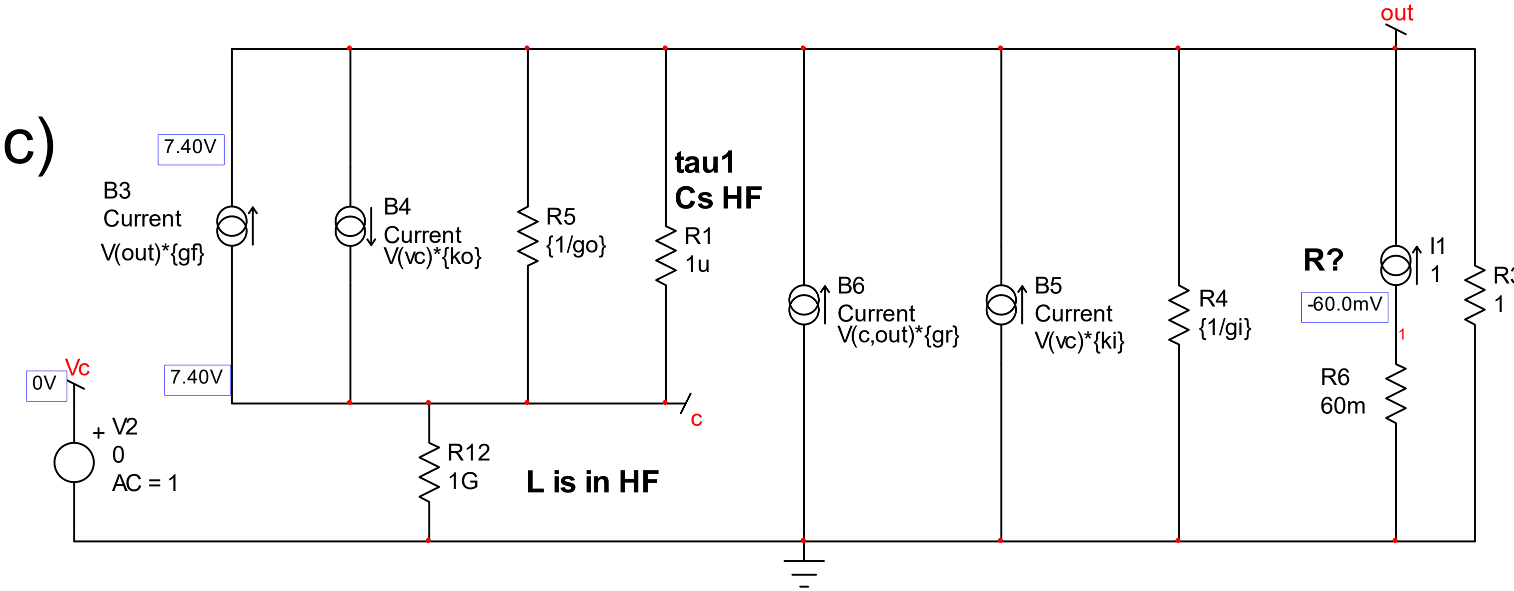
…And a_3 is ?

- For a_3, we multiply by a third time-constant

$$a_3 = \tau_1 \tau_2^1 \tau_3^{1,2}$$ ⟶ Dimension is time³

- What is this new time constant definition?

$\tau_3^{1,2}$ | C_s (HF) L (HF) / C (dc) ↑ R?



Rearrange and Simplify

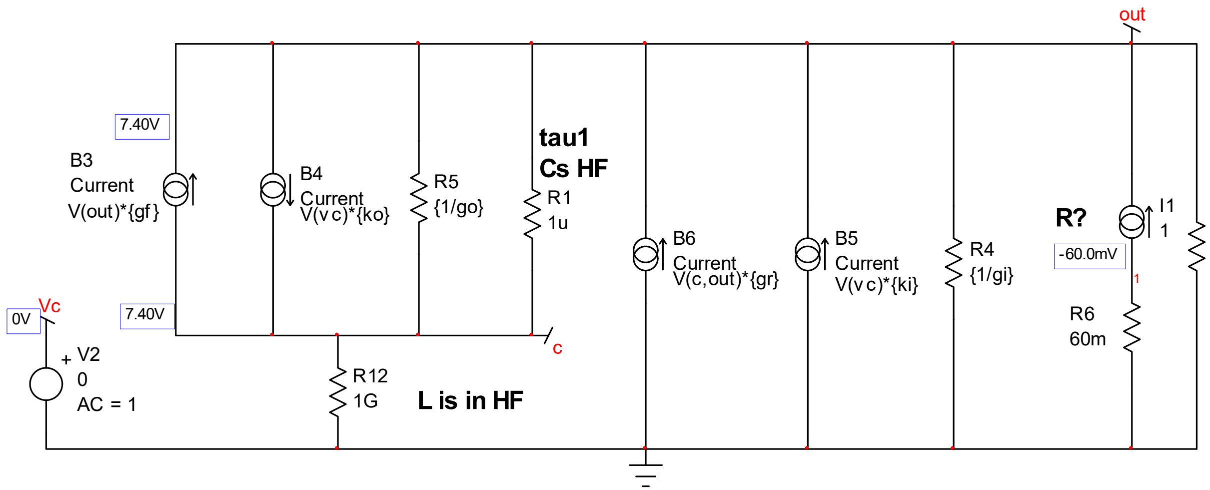
parameters
Fsw =1Meg
Tsw =1/Fsw
L=5u
Cs=1/(L*(Fsw *3.14)^2)
Ri=-50m
Se=0
Vin=2.7
Vout=5

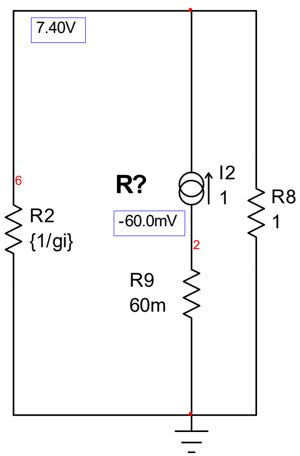
Sn=(Vac/L)*Ri
Sf=(Vap/L)*Ri

Vc=471m
Vac=-Vin
Vap=-Vout
Vcp=(-Vout+Vin)

Ic=(Vc/Ri)-D*Tsw *Se-Vcp*(1-D)*Tsw /(2*L)

D=Vcp/Vap
D'=1-D
ki=D/Ri
gi=D*(gf-Ic/Vap)
gr=(Ic/Vap)-go*D
ko=1/Ri
go=(Tsw /L)*(D'*Se/Sn+0.5-D)
gf=D*go-(D*D')*Tsw /(2*L)





$$\tau_3^{1,2} = \left(\frac{1}{g_i} \parallel r_L + r_C \right) C_{out}$$

Final Denominator Expression

$$a_1 = \tau_1 + \tau_2 + \tau_3$$

$$a_1 = \left(\frac{1}{g_r - g_f + \frac{1}{R_{eq1}}} \right) C_s + \frac{L}{\frac{1 - R_{eq2}\left(g_f - g_o\right)}{g_o + R_{eq2} g_r \left(g_f - g_o\right)}} + \left(\frac{1}{g_r - g_f + \frac{1}{R_{eq1}}} + r_C \right) C_{out}$$

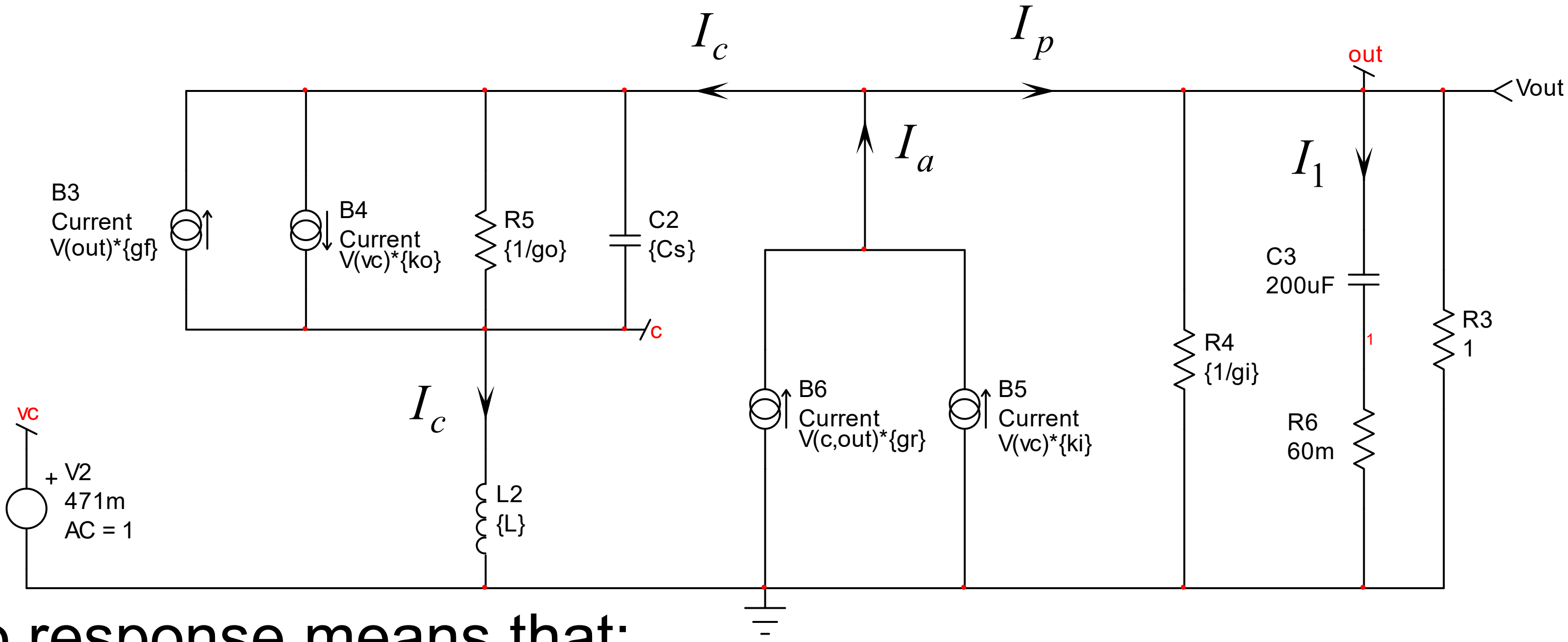
$$a_2 = \tau_1 \tau_2^1 + \tau_1 \tau_3^1 + \tau_2 \tau_3^2 \rightarrow \tau_1 \tau_2^1 + \tau_1 \tau_3^1 + \tau_3 \tau_2^3$$

$$a_2 = \left(\frac{1}{g_r - g_f + \frac{1}{R_{eq1}}} \right) C_s \frac{L}{\frac{1}{\frac{1}{R_1 \| \frac{1}{g_r} \| \frac{1}{g_i}} - g_r}} + \left(\frac{1}{g_r - g_f + \frac{1}{R_{eq1}}} \right) C_s r_C C_{out} + \left(\frac{1}{g_r - g_f + \frac{1}{R_{eq1}}} + r_C \right) C_{out} \frac{L}{\frac{1 - R_{eq2}\left(g_f - g_o\right)}{g_0 + R_{eq2} g_r \left(g_f - g_o\right)}}$$

$$a_3 = \tau_1 \tau_2^1 \tau_3^{12} \qquad a_3 = \left(\frac{1}{g_r - g_f + \frac{1}{R_{eq1}}} \right) C_s \frac{L}{\frac{1}{\frac{1}{R_1 \| \frac{1}{g_r} \| \frac{1}{g_i}} - g_r}} \left(\frac{1}{g_i} \| r_L + r_C \right) C_{out}$$

Find the Zeros

- ❑ A zero means the *excitation* does not generate a *response*

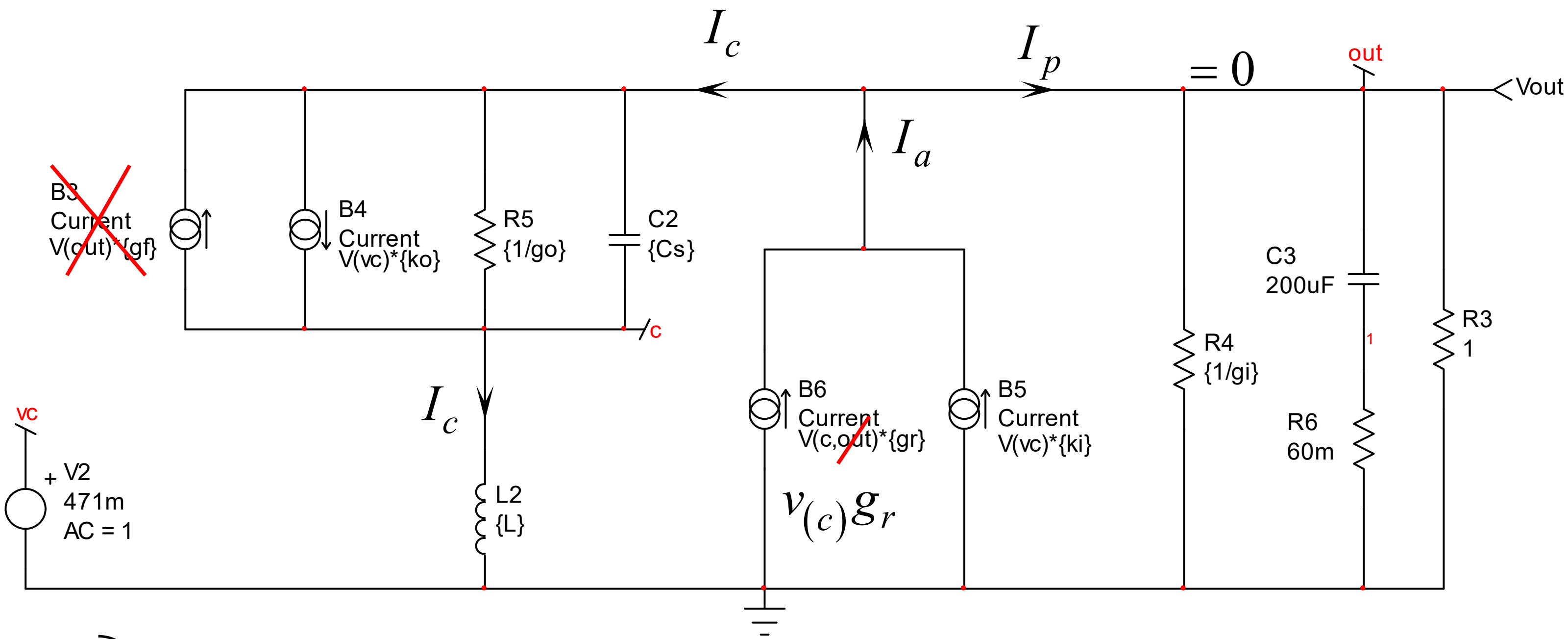


No response means that:

$$I_p = 0 \left\} \begin{array}{l} I_p = I_1 \longrightarrow \text{Short circuit} \longrightarrow r_C + sC_{out} = 0 \longrightarrow s_{z_1} = -\dfrac{1}{r_C C_{out}} \\ I_p = 0 \longrightarrow V_{out} = 0 \end{array} \right.$$

Find the Zeros

❑ A Capitalize on the 0-V response to remove sources



$$I_a = I_p \quad \begin{cases} v_{(c)}g_r + V_c k_i = I_c \\ I_c = V_c k_o - v_{(c)}g_o - v_{(c)}sC_s \\ v_{(c)}g_r + V_c k_i = V_c k_o - v_{(c)}g_o - v_{(c)}sC_s \end{cases}$$

$$\frac{v_{(c)}}{sL} = V_c k_o - v_{(c)}g_o - v_{(c)}sC_s$$

$$v_{(c)} = \frac{V_c k_o}{\frac{1}{sL} + g_o + sC_s}$$

Find The Zeros

- After substituting $v_{(c)}$ into the equation rearrange it

$$\frac{V_c\left(k_i + Lg_o k_i s + Lg_r k_o s + C_s L k_i s^2\right)}{1 + sLg_o + s^2 C_s L} = \frac{V_c k_o}{1 + sLg_o + s^2 C_s L}$$

$$\Rightarrow \quad k_i + Lg_o k_i s + Lg_r k_o s + C_s L k_i s^2 = k_o$$

$$\left(k_i - k_o\right)\left(1 + sL\left(\frac{g_o k_i + g_r k_o}{k_i - k_o}\right) + s^2 \frac{C_s L k_i}{k_i - k_o}\right) = 0$$

$$\Rightarrow \quad 1 + sL\left(\frac{g_o k_i + g_r k_o}{k_i - k_o}\right) + s^2 \frac{C_s L k_i}{k_i - k_o} = 0$$

For low Q values, the 2nd order polynomial can expressed as

$$1 + a_1 s + a_2 s^2 \approx \left(1 + a_1 s\right)\left(1 + \frac{a_2}{a_1} s\right)$$

Find the Zeros – Final Lap!

- The first zero is a right half-plane zero

$$1+sL\left(\frac{g_o k_i + g_r k_o}{k_i - k_o}\right) = 1 - \frac{s}{s_{z_1}} \qquad s_{z_1} = \frac{k_i - k_o}{L\left(g_o k_i + g_r k_o\right)} \Rightarrow \omega_{z_1} \approx \frac{(1-D)^2 R_L}{L}$$

Negative value!

- The second zero is given by the capacitor ESR

$$1 + s r_C C_{out} = 0 \Rightarrow \omega_{z_2} \approx \frac{(1-D)^2 R_L}{L}$$

- The third zero is located at high frequency

$$1 + s\frac{C_s k_i}{g_o k_i + g_r k_o} = 1 + \frac{s}{s_{z_3}} \qquad s_{z_3} = \frac{g_o k_i + g_r k_o}{C_s k_i} \Rightarrow \text{Neglect it}$$

$$\longrightarrow f_{z2} := \frac{\omega_{z2}}{2\pi} = 3.16 \times 10^4 \text{ kHz}$$

V. Vorpérian,"Analytical Methods in Power Electronics", 3-day course in Toulouse,2004

Arranging the Transfer Function

- The denominator has to be rearranged a little bit…

$$D(s)=1+b_1 s+b_2 s^2+b_3 s^3$$

- The term $1+b_1 s$ dominates at low frequency

$$1+b_1 s+b_2 s^2+b_3 s^3 \approx (1+b_1 s)\left(1+\frac{b_2}{b_1}s+\frac{b_3}{b_1}s^2\right)$$

$$(1+b_1 s)\left(1+\frac{b_2}{b_1}s+\frac{b_3}{b_1}s^2\right) \approx \left(1+\frac{s}{\omega_{p_1}}\right)\frac{1}{1+\frac{s}{\omega_n Q_p}+\left(\frac{s}{\omega_n}\right)^2}$$

$$\omega_{p_1} \approx \frac{\frac{2}{R_L}+\frac{T_{sw}}{LM^3}\left(1+\frac{S_a}{S_n}\right)}{C_{out}}$$

$$Q_p=\frac{1}{\pi(m_c D'-0.5)} \qquad \omega_n=\frac{\pi}{T_{sw}} \qquad m_c=1+\frac{S_a}{S_n}$$

Dc gain H_0

$$H_0 \approx \frac{R_L}{R_i}\frac{1}{2M+\frac{R_L T_{sw}}{LM^2}\left(\frac{1}{2}+\frac{S_a}{S_n}\right)}$$

Final Expression

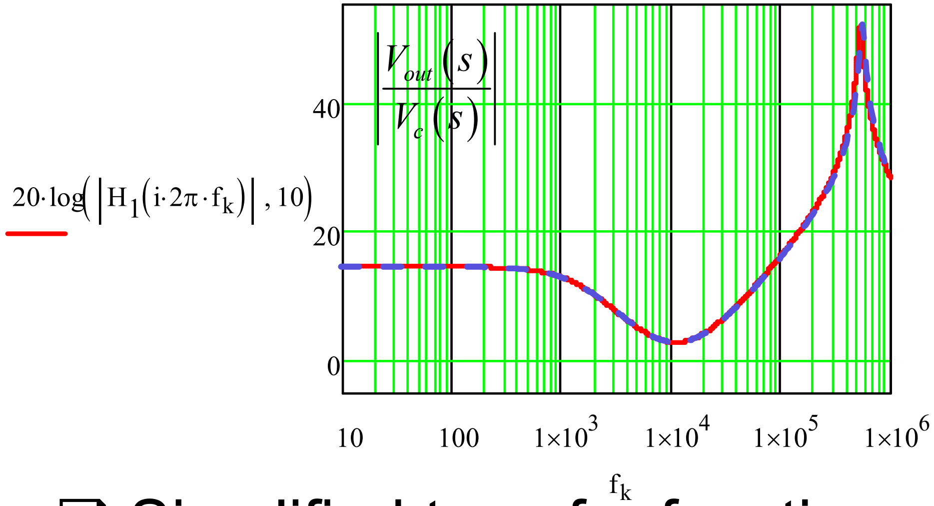
❑ The transfer function of a CCM Boost Converter in CM is

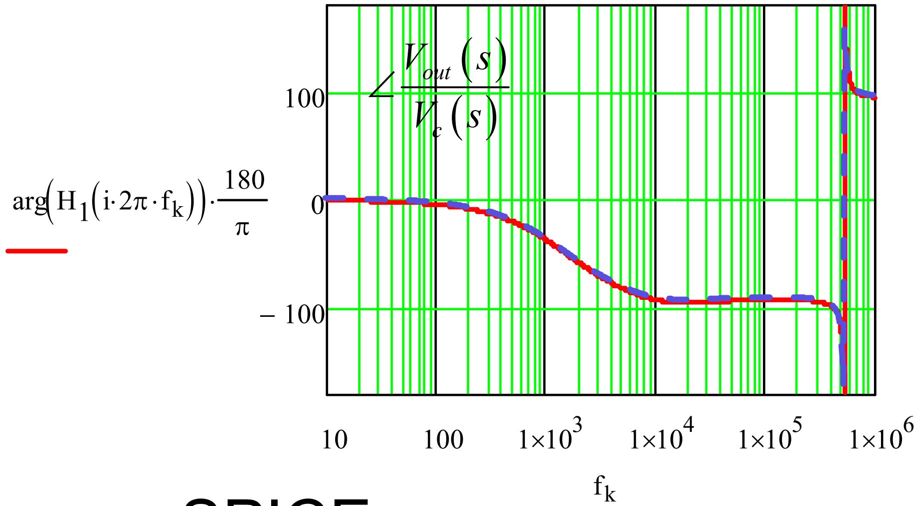
$$\frac{V_{out}(s)}{V_c(s)} \approx \frac{R_L}{R_i} \frac{1}{2M + \frac{R_L T_{sw}}{LM^2}\left(\frac{1}{2}+\frac{S_a}{S_n}\right)} \frac{\left(1 - s\frac{L}{(1-D)^2 R_L}\right)\left(1+sr_C C_{out}\right)}{1 + s\frac{C_{out}}{\frac{2}{R_L}+\frac{T_{sw}}{LM^3}\left(1+\frac{S_a}{S_n}\right)}} \frac{1}{1+\frac{s}{\omega_n Q_p}+\left(\frac{s}{\omega_n}\right)^2}$$

S_a is the external ramp compensation to damp subharmonic oscillations

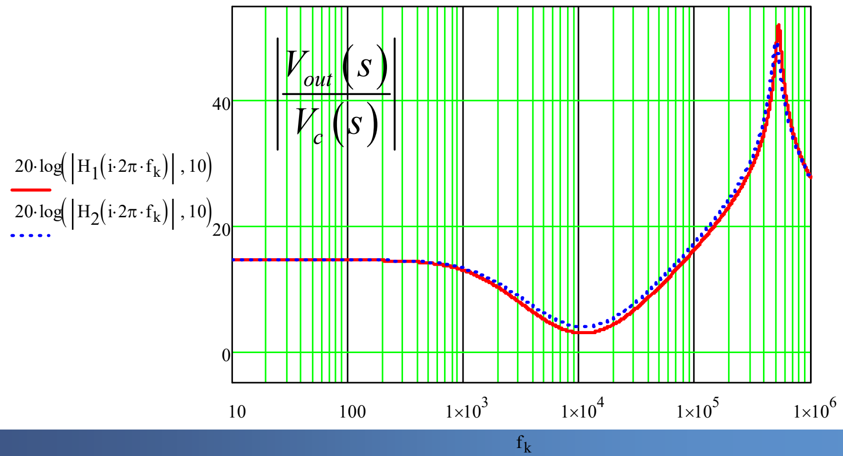
Check SPICE Versus Mathcad

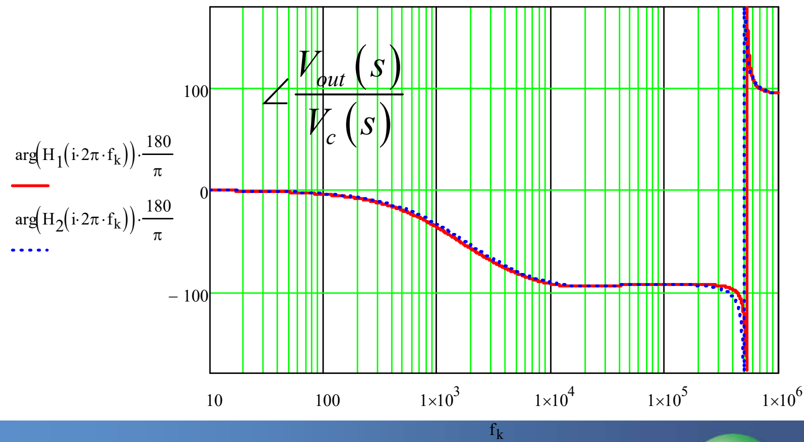
- ❑ Full-featured transfer functions versus SPICE





- ❑ Simplified transfer functions versus SPICE





Course Agenda

- ❑ The PWW Switch Concept
- ❑ Small-Analysis in Continuous Conduction Mode
- ❑ **Small-Signal Response in Discontinuous Mode**
- ❑ EMI Filter Output Impedance
- ❑ Cascaded Converters Operation

The CM Boost Converter in DCM

- To check the operating mode, calculate the critical load

$$R_{crit} = \frac{2F_{sw}L{V_{out}}^2}{{V_{in}}^2\left(1-\frac{V_{in}}{V_{out}}\right)}$$

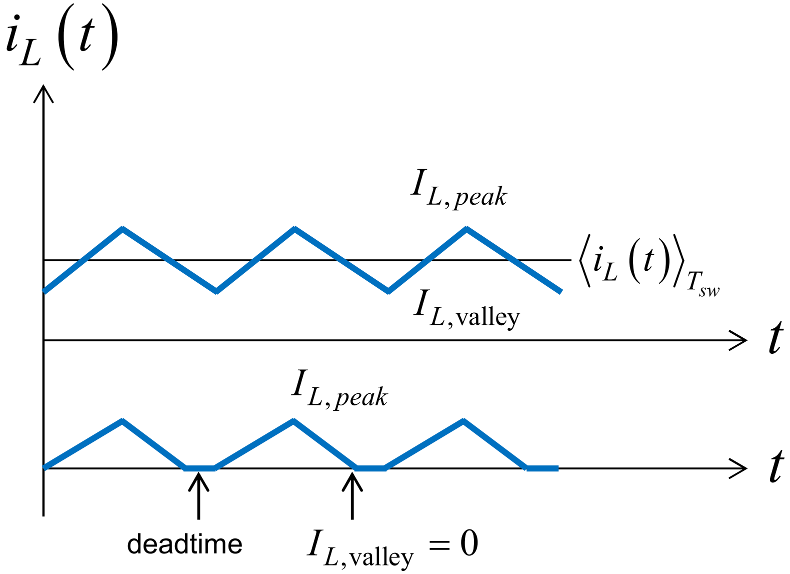
$F_{sw} = 1\text{ MHz}$

$L = 5\ \mu\text{H}$

$V_{in} = 2.7\text{ V}$

$V_{out} = 5\text{ V}$

$\longrightarrow R_{crit} = 74.7\ \Omega$



CCM, $R_L < R_{crit}$

DCM, $R_L > R_{crit}$

Transfer Function of DCM CM Boost

- The converter is a first-order converter at low frequency

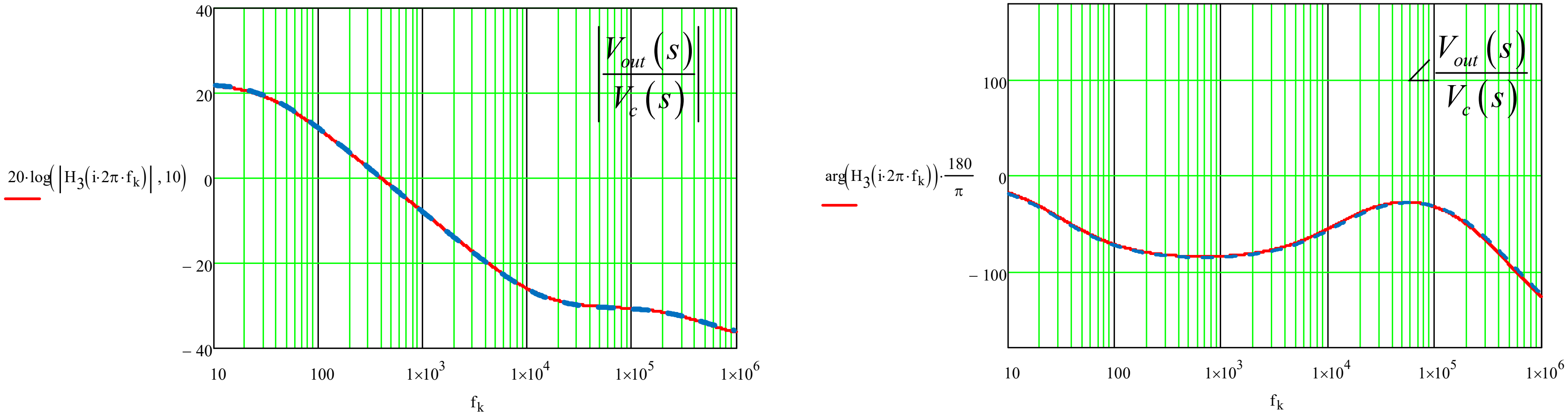
$$H(s)=\frac{1}{S_n m_c T_{sw}}\frac{2V_{out}}{D}\frac{M-1}{2M-1}\frac{\left(1+\frac{s}{\omega_{z_1}}\right)\left(1-\frac{s}{\omega_{z_2}}\right)}{\left(1+\frac{s}{\omega_{p_1}}\right)\left(1+\frac{s}{\omega_{p_2}}\right)}$$

$$S_n=\frac{V_{in}}{L}R_i \qquad \omega_{z_1}=\frac{1}{r_C C_{out}} \qquad \omega_{z_2}=\frac{R_{load}}{M^2 L} \qquad \omega_{p_1}=\frac{1}{R_L C_{out}}\frac{2M-1}{M-1}$$

$$\omega_{p_2}=2F_{sw}\left(\frac{1-\frac{1}{M}}{D}\right)^2 \qquad m_c=1+\frac{S_a}{S_n}$$

The Ac Response of the DCM CM Boost

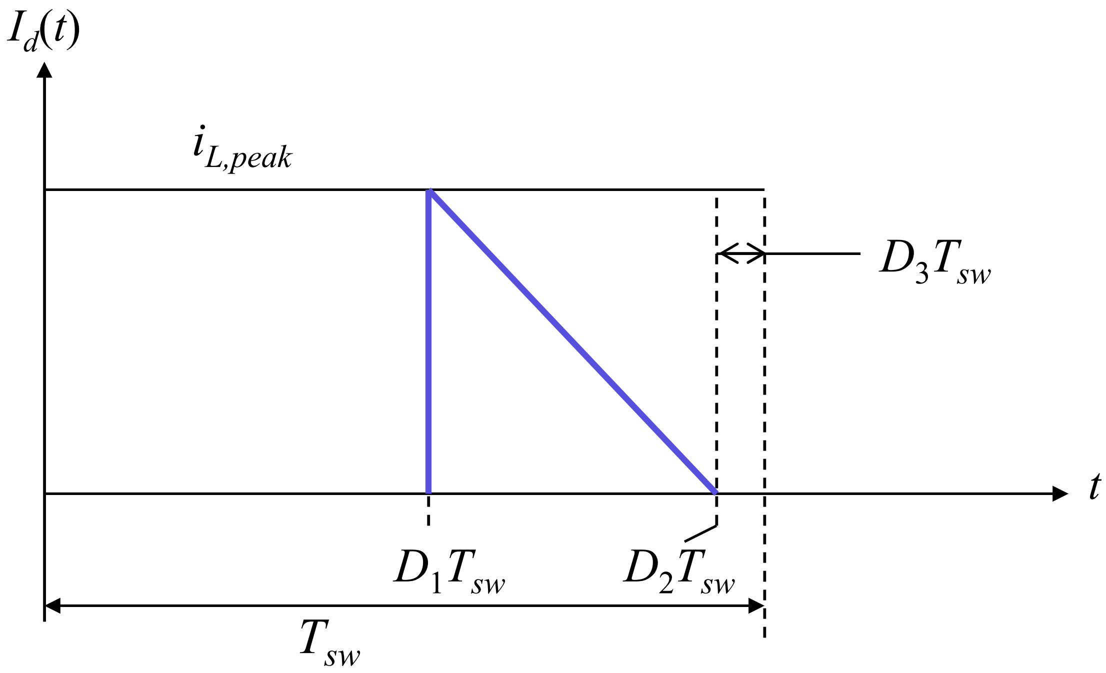
- ❑ A RHPZ is still present in DCM with a high-frequency pole



SPICE simulation versus Mathcad

Why a RHP Zero in DCM?

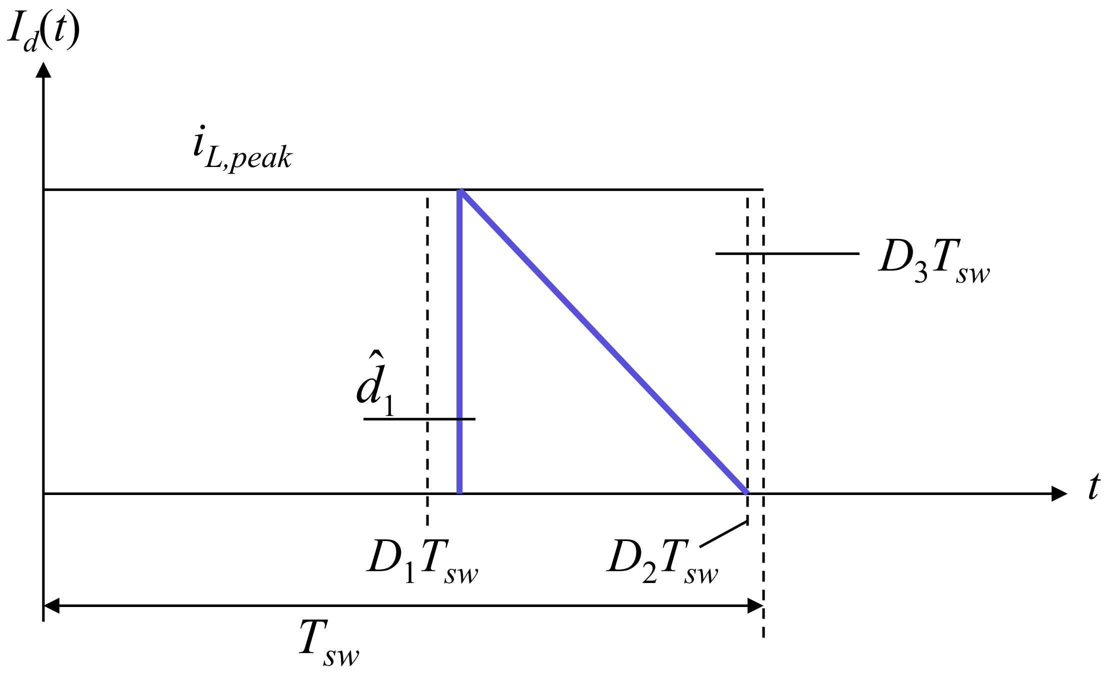
- The RHP Zero is present in CCM and is still there in DCM!



- When D_1 increases, $[D_1,D_2]$ stays constant but D_3 shrinks

Why a RHP Zero in DCM?

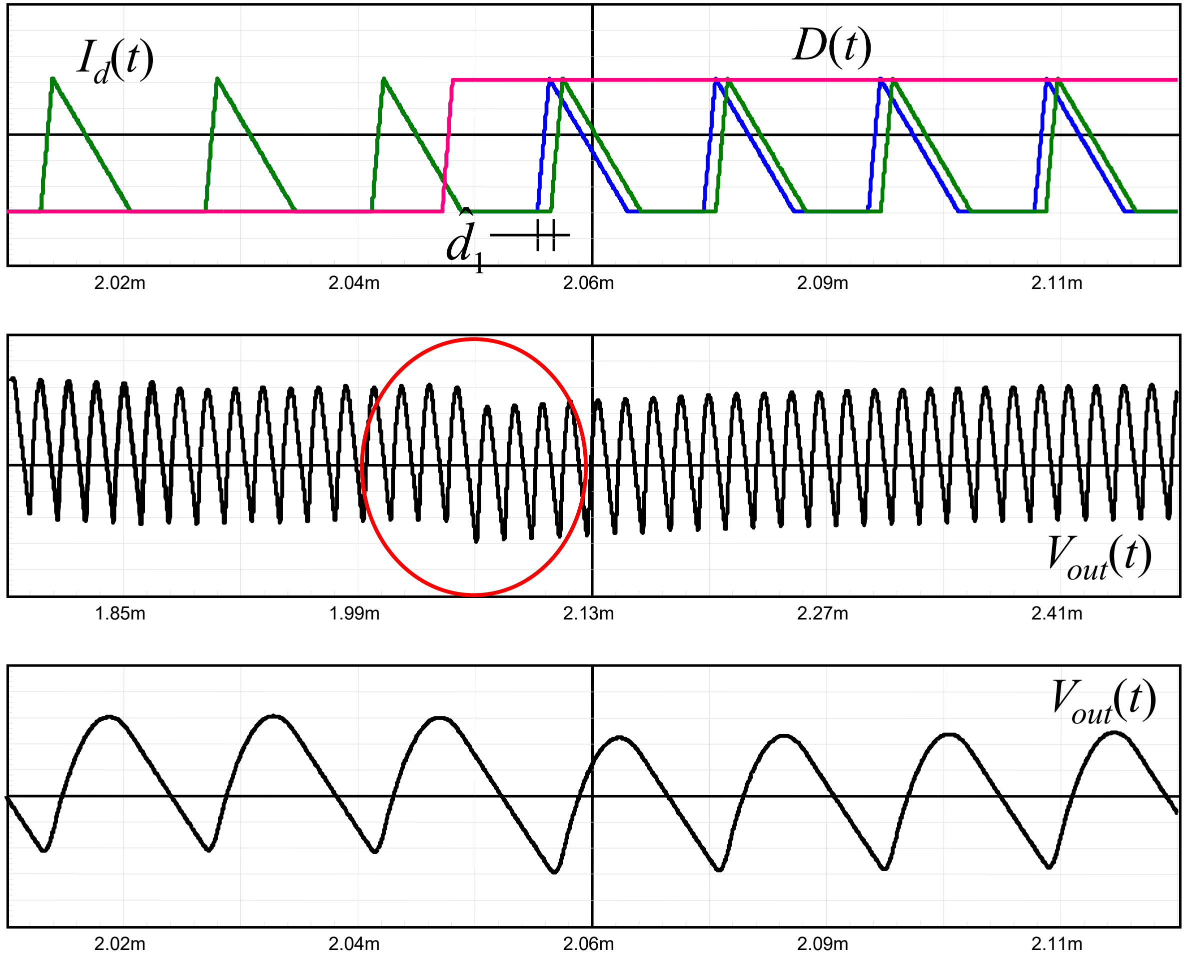
- The triangle is simply shifted to the right by $\hat{d}_1$



- The capacitor refueling time is delayed and a drop occurs

The Refueling Time is Shifted

- If D increases, the diode current is delayed by $\hat{d}_1$



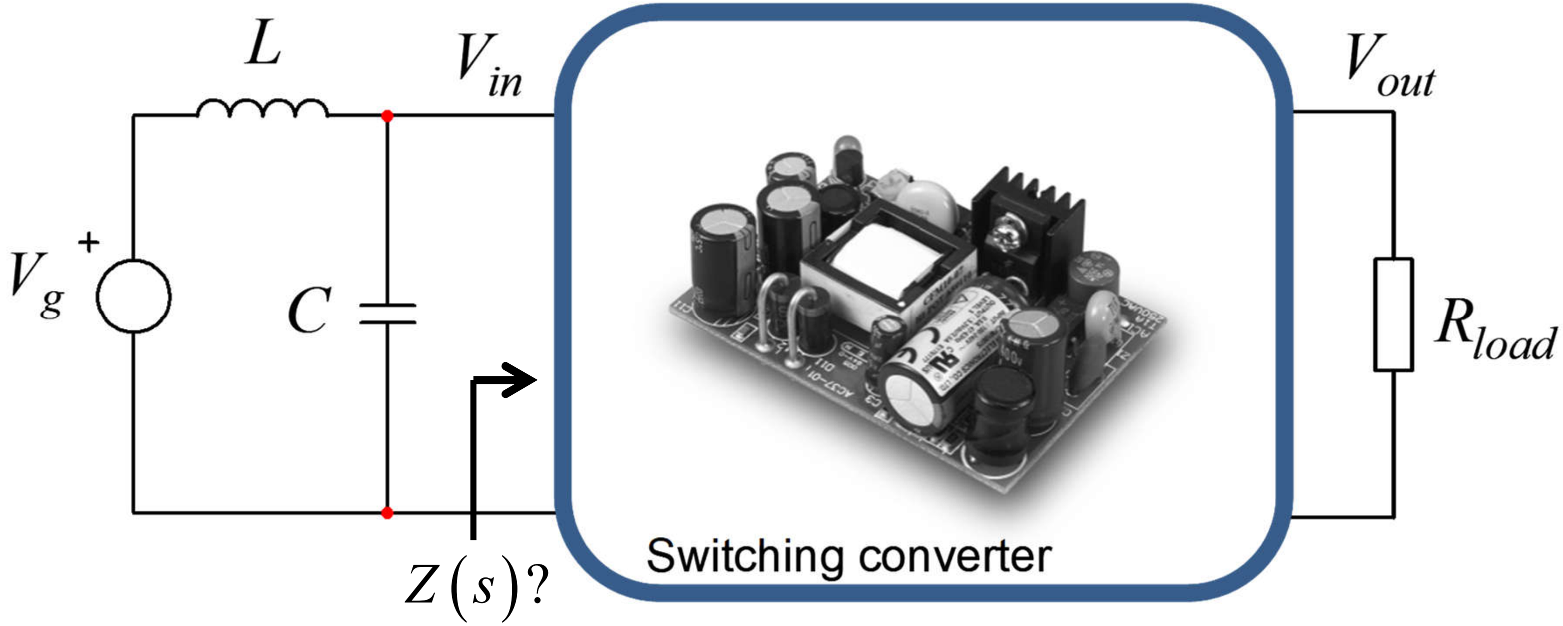
Simulation, no peak increase

Course Agenda

- ❑ The PWW Switch Concept
- ❑ Small-Analysis in Continuous Conduction Mode
- ❑ Small-Signal Response in Discontinuous Mode
- ❑ **EMI Filter Output Impedance**
- ❑ Cascaded Converters Operation

EMI Filter Interaction

- A LC filter is included to prevent the dc line pollution



- What load does the converter offer?

C. Basso, “Designing Control Loops for Linear and Switching Power Supplies”, Artech House, 2012

A Negative Incremental Resistance

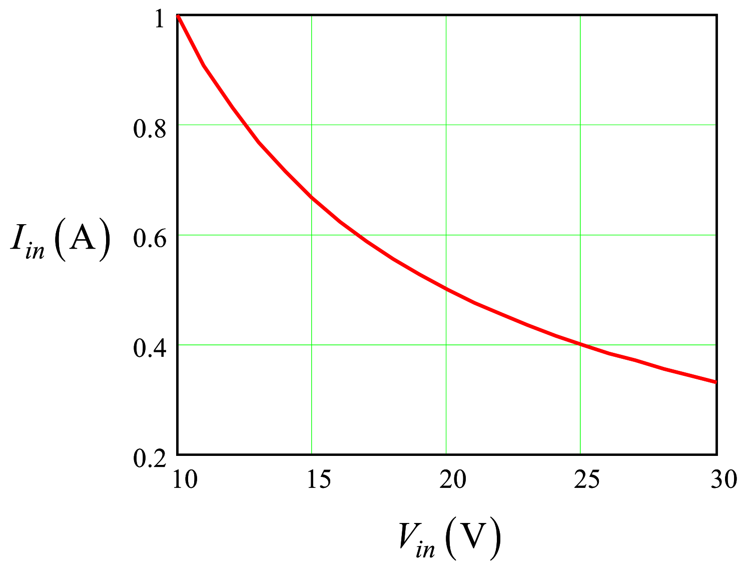
- Assume a 100%-efficient converter:

$$P_{out} = P_{in} \longrightarrow I_{in}V_{in} = I_{out}V_{out}$$

- In closed-loop operation, P_{out} is constant

$$\longrightarrow I_{in}\left(V_{in}\right) = \frac{P_{out}}{V_{in}}$$

- For a constant P_{out} , if V_{in} increases, I_{in} drops

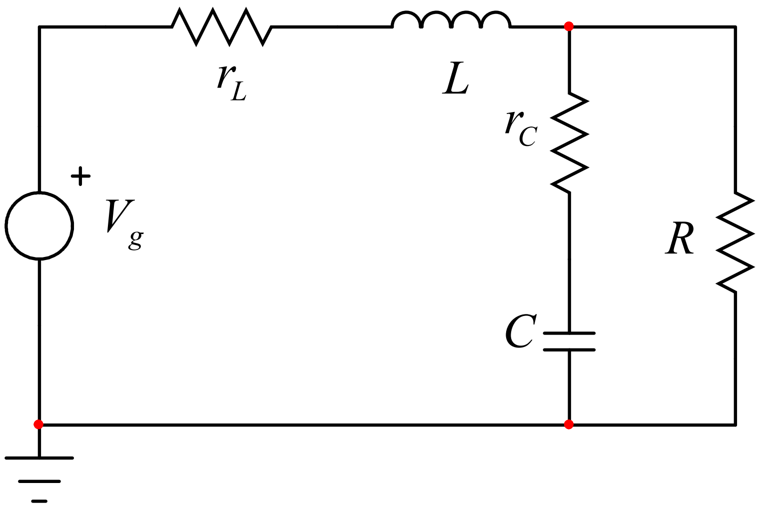


$$\frac{dI_{in}\left(V_{in}\right)}{dV_{in}} = \frac{d\left(\frac{P_{out}}{V_{in}}\right)}{dV_{in}} = -\frac{P_{out}}{V_{in}^{\ 2}}$$

The incremental input resistance is negative

A Simple LC Filter

- The low-pass filter is built with a L and C components:



$$T(s) = \frac{1 + s/\omega_{z_1}}{1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2}$$

$$\omega_{z_1} = \frac{1}{r_C C}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \sqrt{\frac{r_L + R}{r_C + R}}$$

$$Q = \frac{LC\omega_0 (r_C + R)}{L + C(r_L r_C + r_L R + r_C R)} \approx R_{load} \sqrt{\frac{C}{L}}$$

If R is || with -R

$$Q = \frac{R \cdot (-R)}{R - R} \sqrt{\frac{C}{L}} \to \infty$$

A negative resistance cancels losses: poles become imaginary

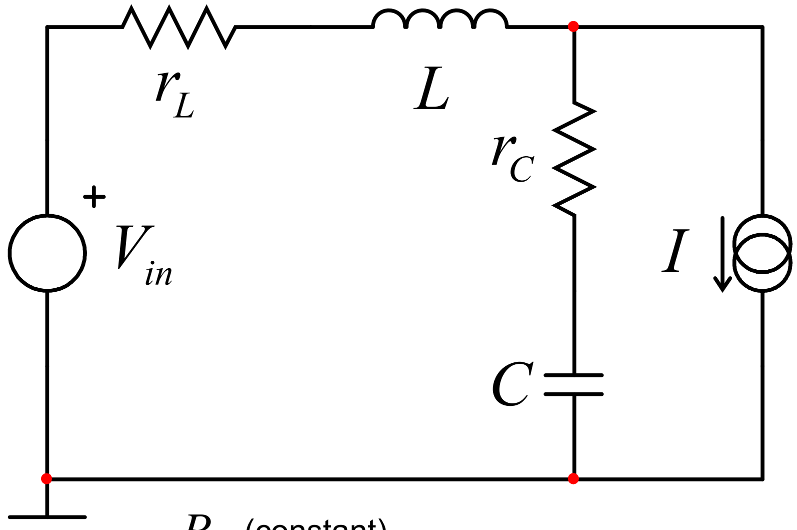
A Negative Impedance Oscillator

- If losses are compensated, the damping factor is zero

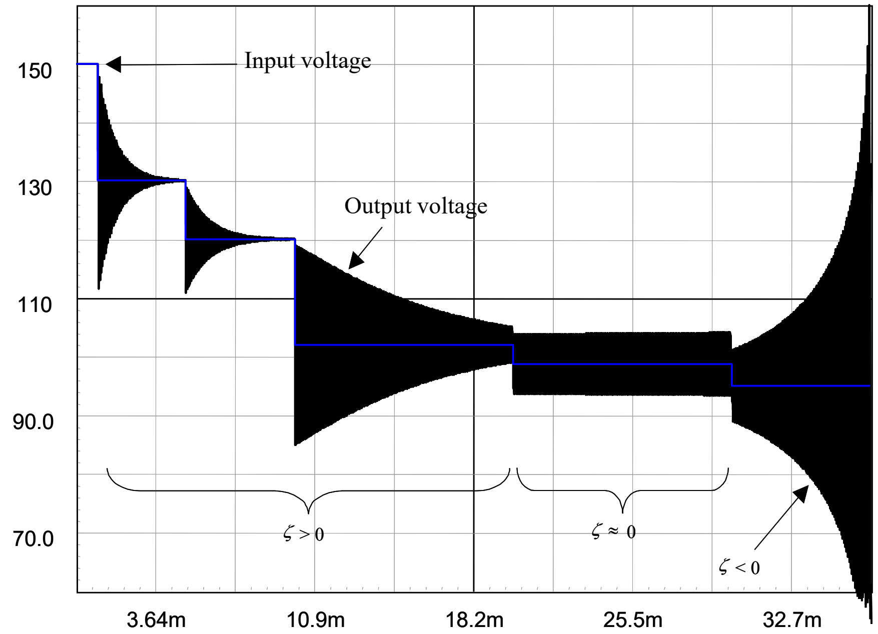
$$T(s) = \frac{1}{\frac{s^2}{\omega_0^2} + \frac{s}{\omega_0 Q} + 1} \qquad Q = \frac{1}{2\zeta}$$

→ If ohmic losses are gone, the damping factor is zero, Q is infinite.

- Without precautions, instability can happen!

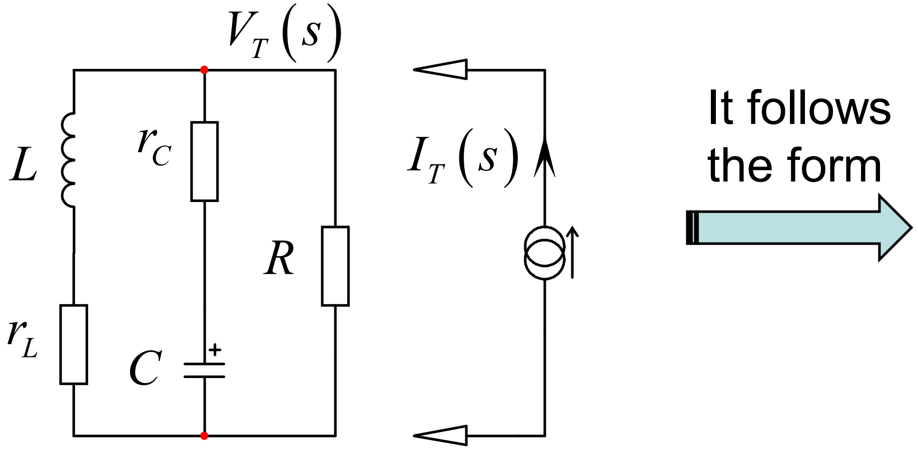


$I = \frac{P_{out}}{V_{out}}$ (constant) → Negative resistance



Filter Output Impedance

- ❑ What is the output impedance of an LC filter?



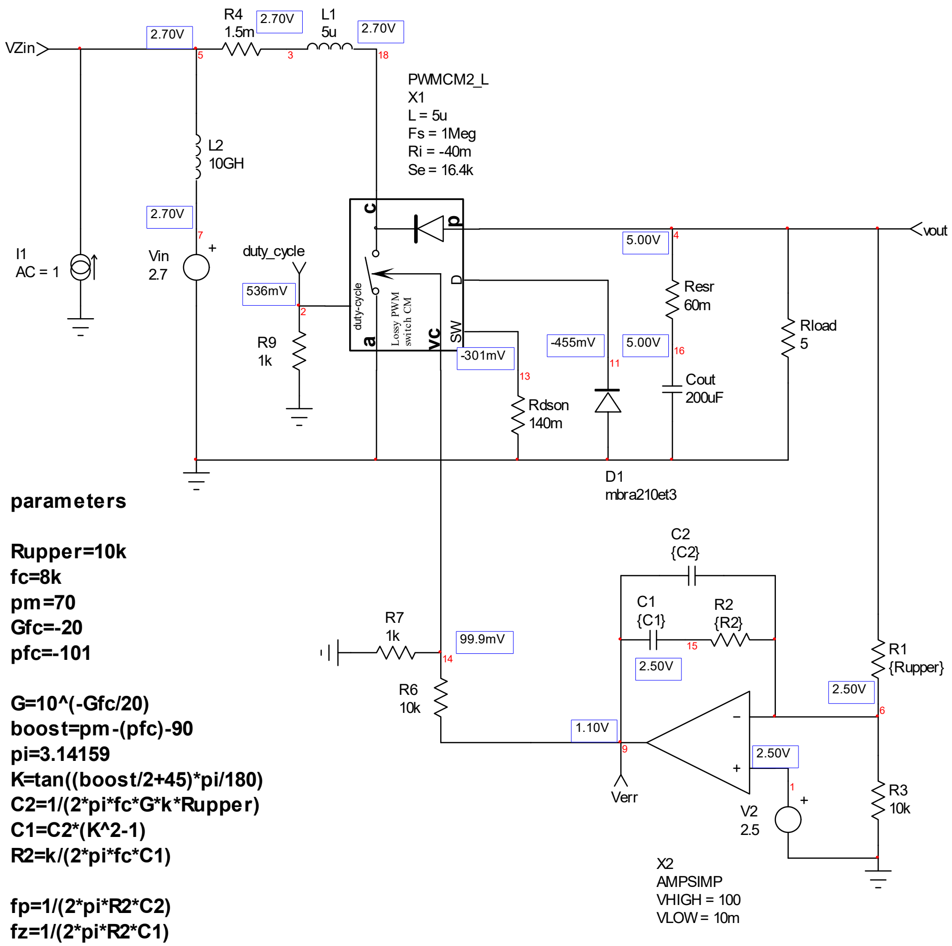
$$Z_{out}(s) = R_0 \frac{N(s)}{D(s)}$$

Z_{out}: Ω, R_0: Ω, $\frac{N(s)}{D(s)}$: No dimension

$$Z_{out}(s) = (r_L \,||\, R_{load}) \frac{\left(1 + s\frac{L}{r_L}\right)(1 + s r_C C_{out})}{1 + s\left(\frac{L}{r_L + R_{load}} + C\left[(r_L \,||\, R_{load}) + r_C\right]\right) + s^2\left(LC\frac{r_C + R_{load}}{r_L + R_{load}}\right)}$$

Negative Resistance at Low Frequency

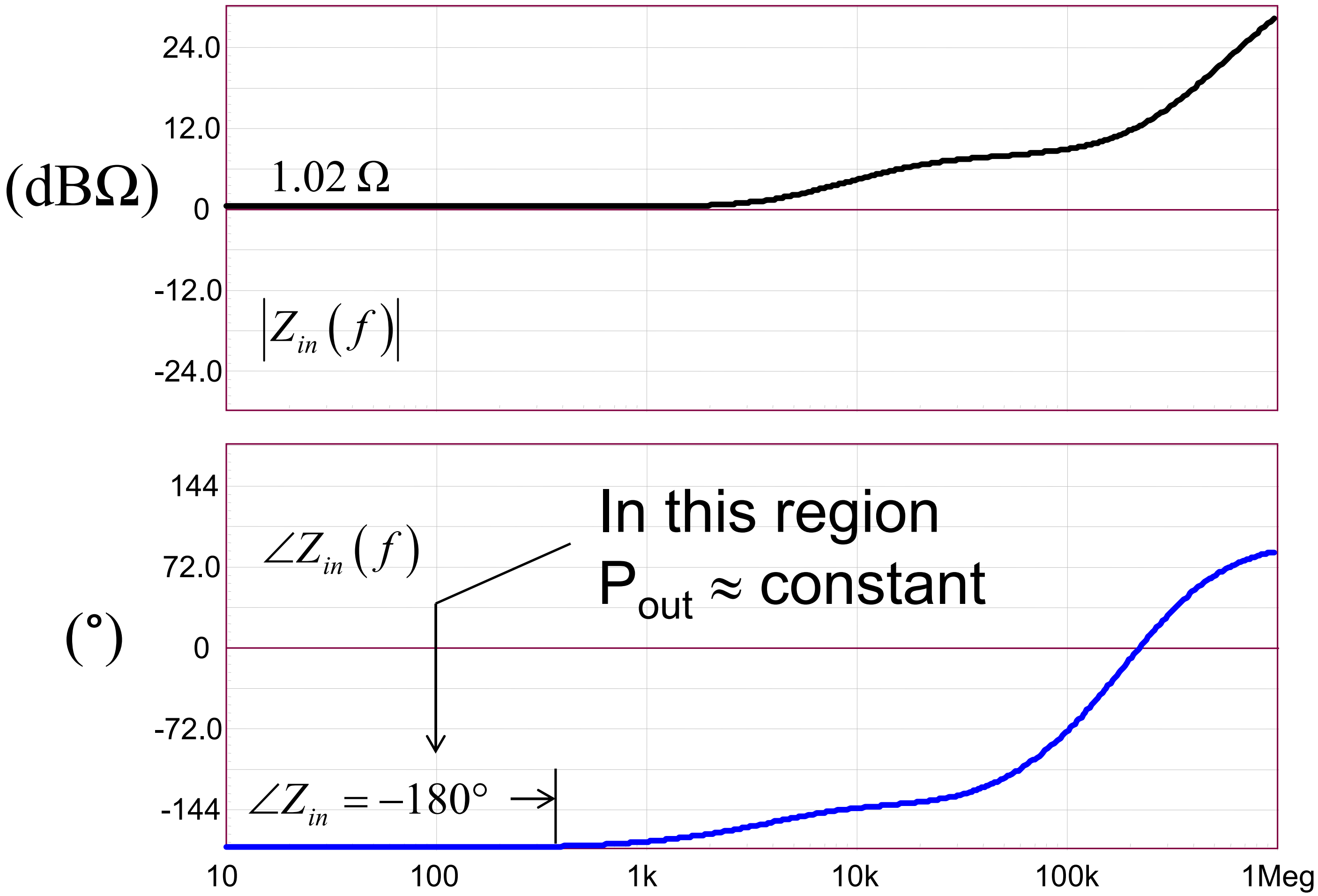
❑ Neg. resistance exists because of feedback (P_{out} = constant)



SwitchMode Power Supplies: SPICE Simulations and Practical Designs
Christophe Basso - McGraw-Hill, 2014

Negative Resistance at Low Frequency

- The resistance is truly negative up to 200 Hz



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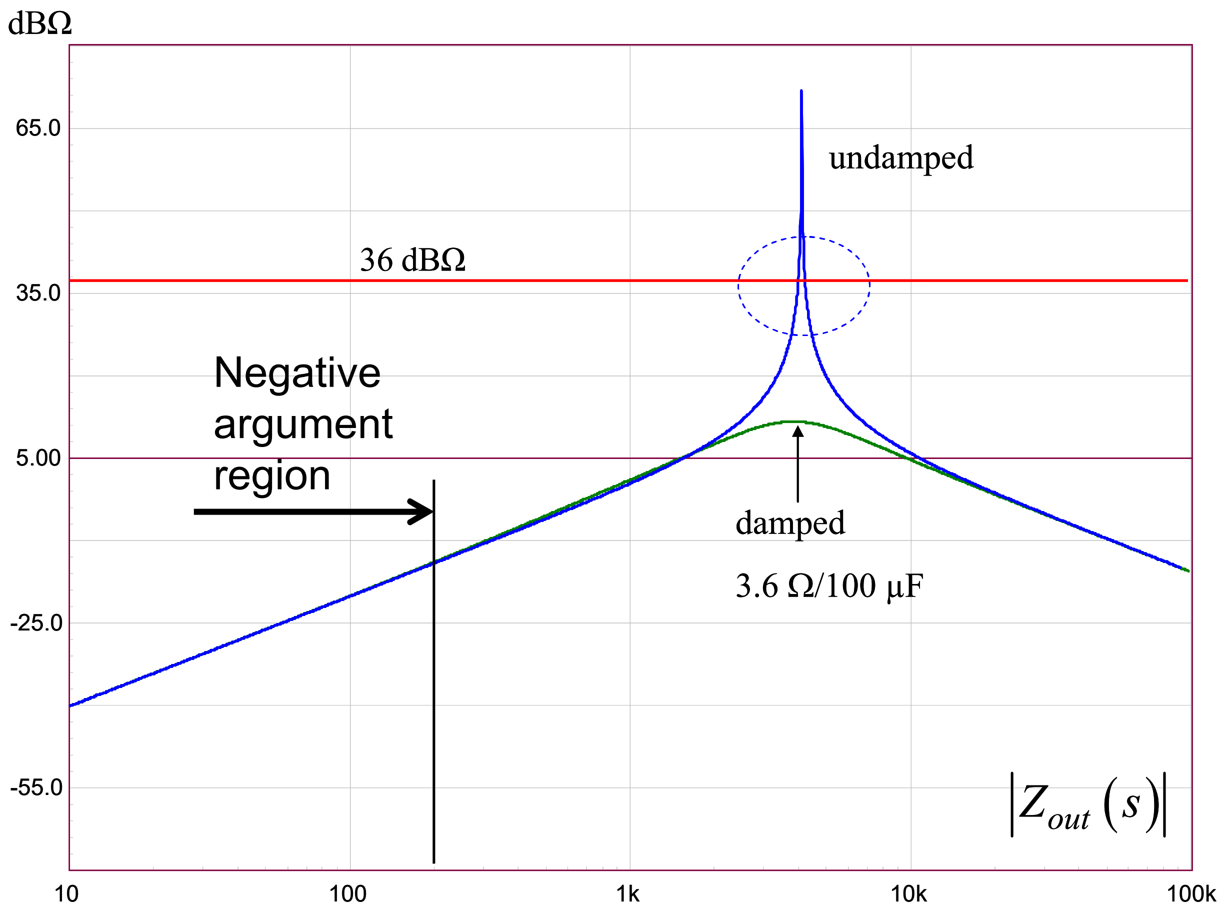
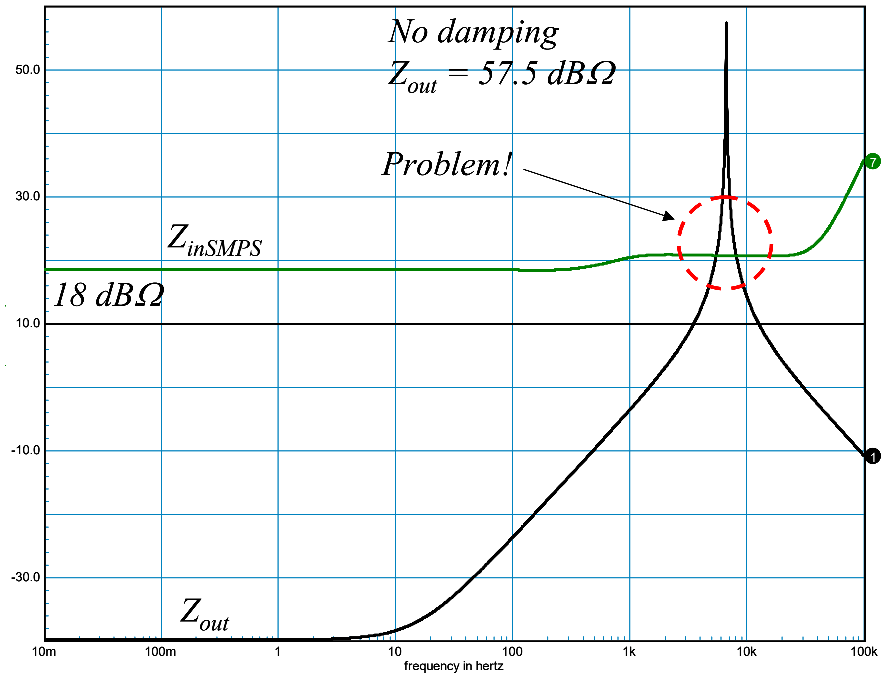
Negative Resistance is One Part Only

- ❑ Negative resistance problems occur only when $\angle Z_{in} = -180°$
- ❑ Beyond this region, instability is linked to gain degradation

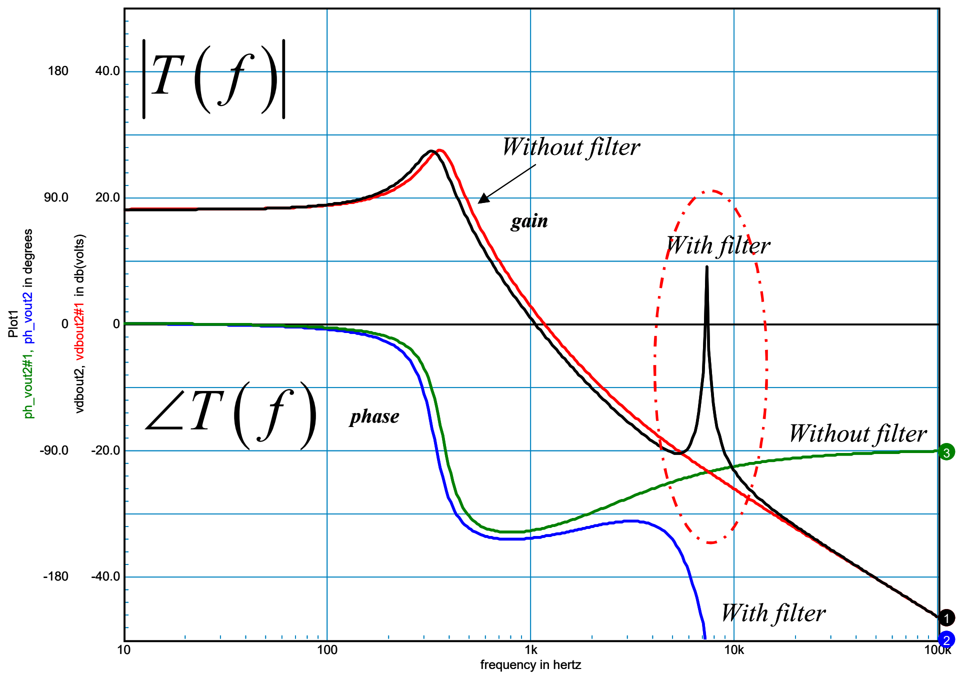


Open-Loop Gain Degradation

- ❑ Here the EMI filter peaks at high frequency
- ❑ The open-loop is severely affected and stability is at stake



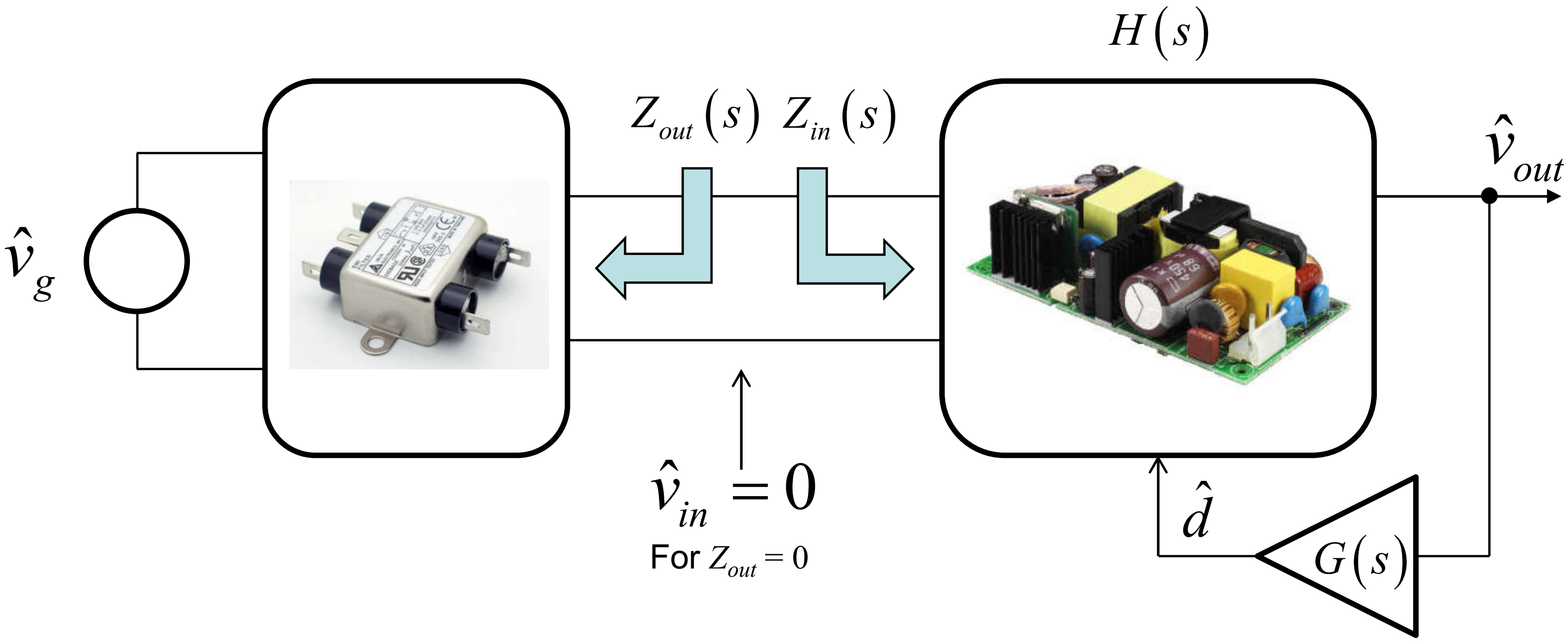
Filter output impedance
versus closed-loop buck input impedance.



Buck open-loop gain with
and without input filter.

Evaluating How Loop Gain is Modified

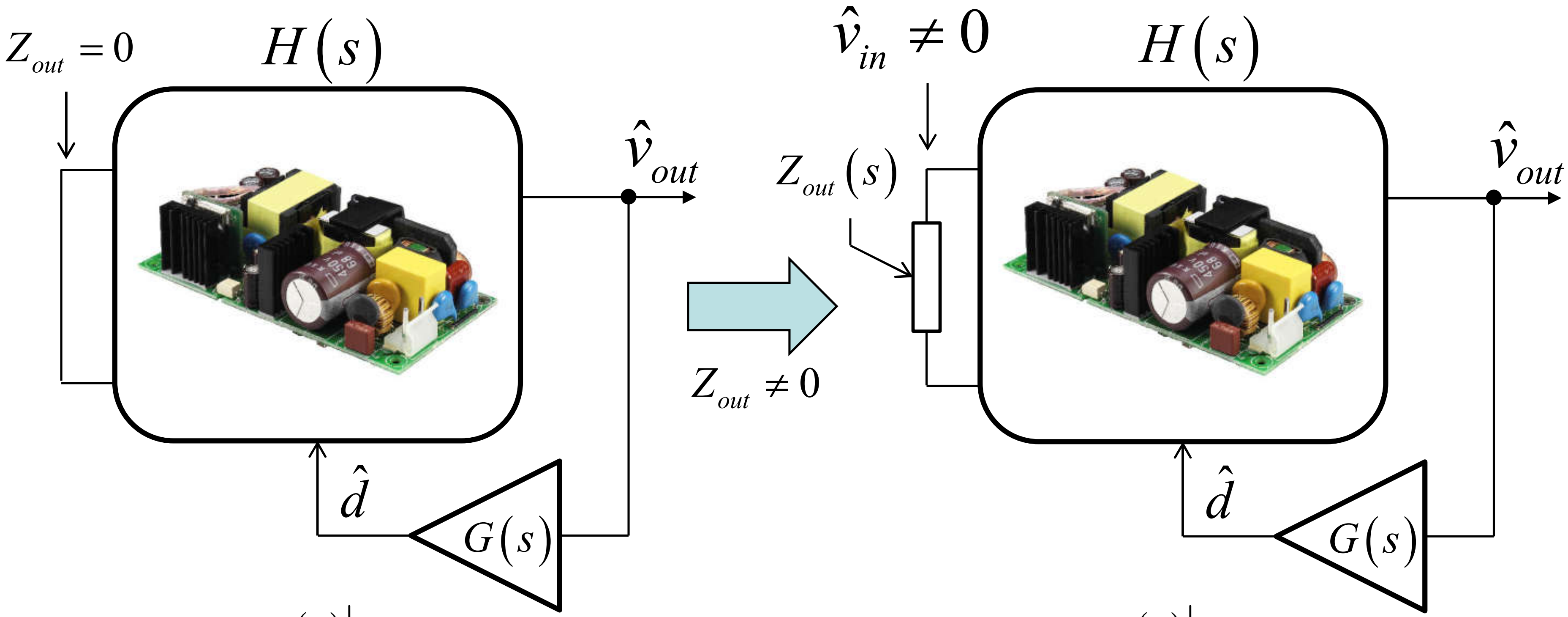
- The converter transfer function is evaluated at $Z_{out} = 0$



$$T(s) = H(s)G(s) = \left.\frac{V_{out}(s)}{D(s)}\right|_{V_{in}(s)=0}$$

Considering the Filter Output Impedance

- In reality, the EMI output impedance makes $Z_{out}(s) \neq 0$
- The converter ac input voltage is no longer zero

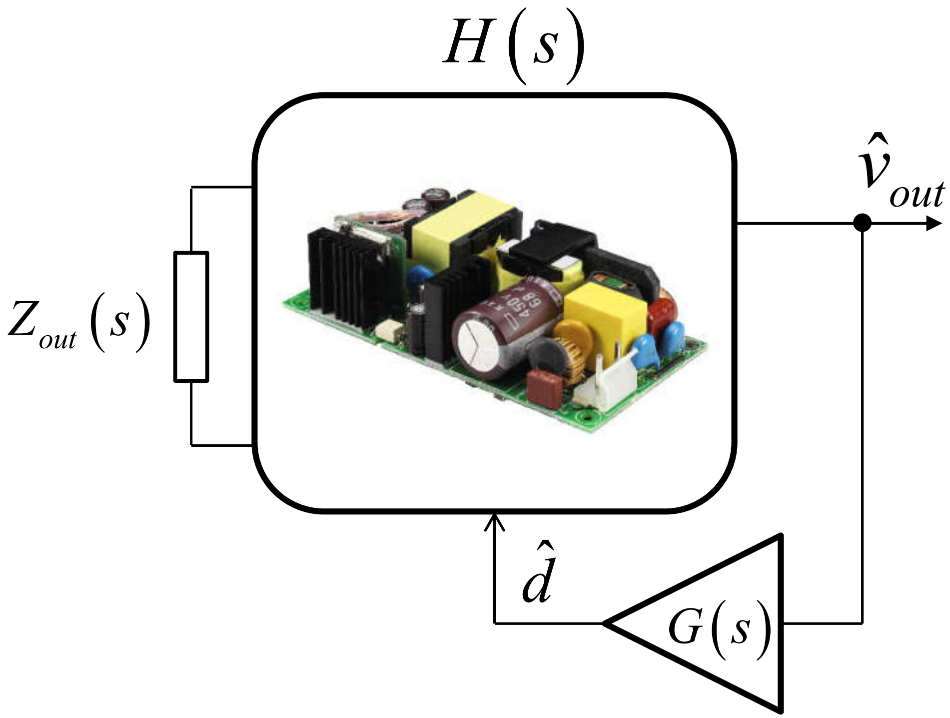


$$T(s) = \left.\frac{V_{out}(s)}{D(s)}\right|_{V_{in}(s)=0}$$

$$T(s) = \left.\frac{V_{out}(s)}{D(s)}\right|_{V_{in}(s)\neq 0}$$

Extra Element Theorem to Help

❑ It can be shown how an EMI affects the open-loop gain:



$$T(s) = \left.\frac{V_{out}(s)}{D(s)}\right|_{Z_{out}(s)=0}$$ Gain without the filter

$Z_{out}(s)$ Filter output impedance

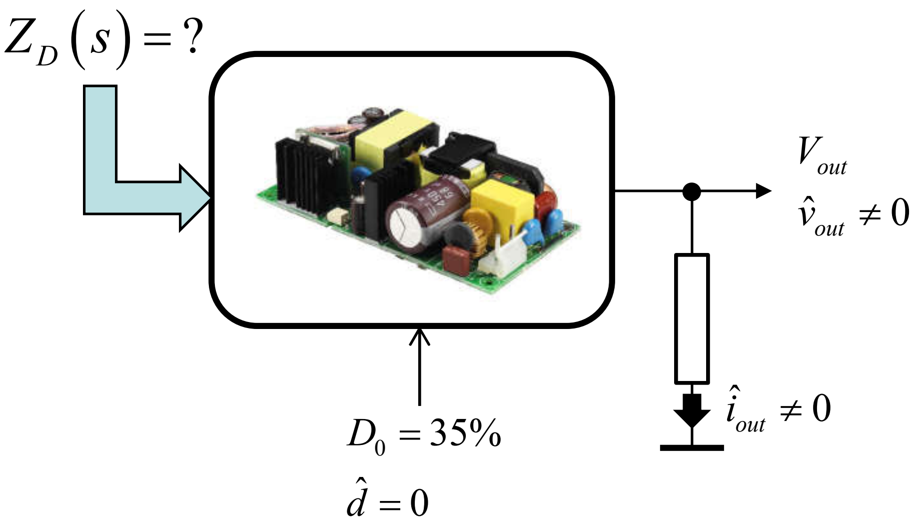
$$T(s) = \left.\frac{V_{out}(s)}{D(s)}\right|_{Z_{out}(s)\neq 0} = \left.\frac{V_{out}(s)}{D(s)}\right|_{Z_{out}(s)=0} \frac{1+\dfrac{Z_{out}(s)}{Z_N(s)}}{1+\dfrac{Z_{out}(s)}{Z_D(s)}}$$

What are Z_D and Z_N?

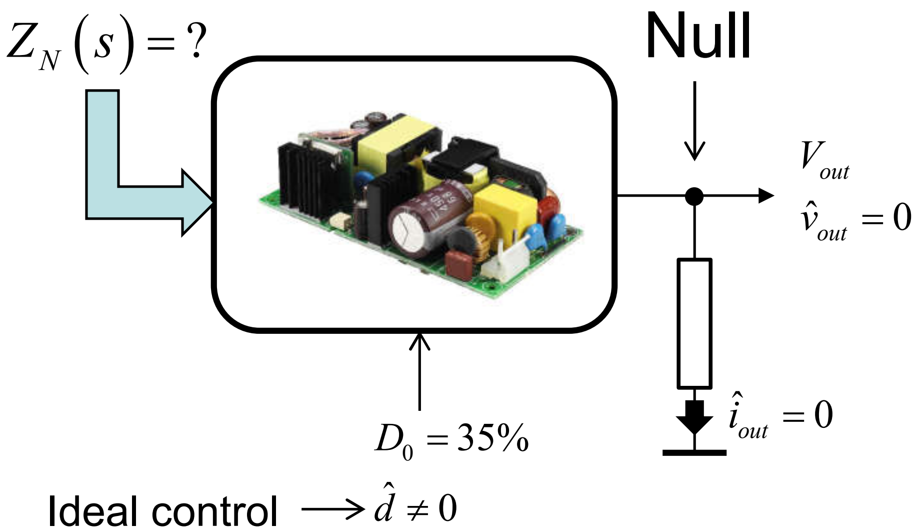
- Z_D and Z_N come from the Extra Element Theorem, EET

$$Z_D(s) = Z_i(s)\Big|_{D(s)=0}$$ ⇒ Open-loop input impedance

$$Z_N(s) = Z_i(s)\Big|_{\hat{v}_{out}=0}$$ ⇒ Input impedance for a nulled output



Open-loop input impedance



Input impedance for $V_{out}(s) = 0$

What are Z_D and Z_N for a Boost Converter?

- These values have already been derived

$$Z_N(s) = -D'^2 R\left(1 - \frac{sL}{D'^2 R}\right)$$

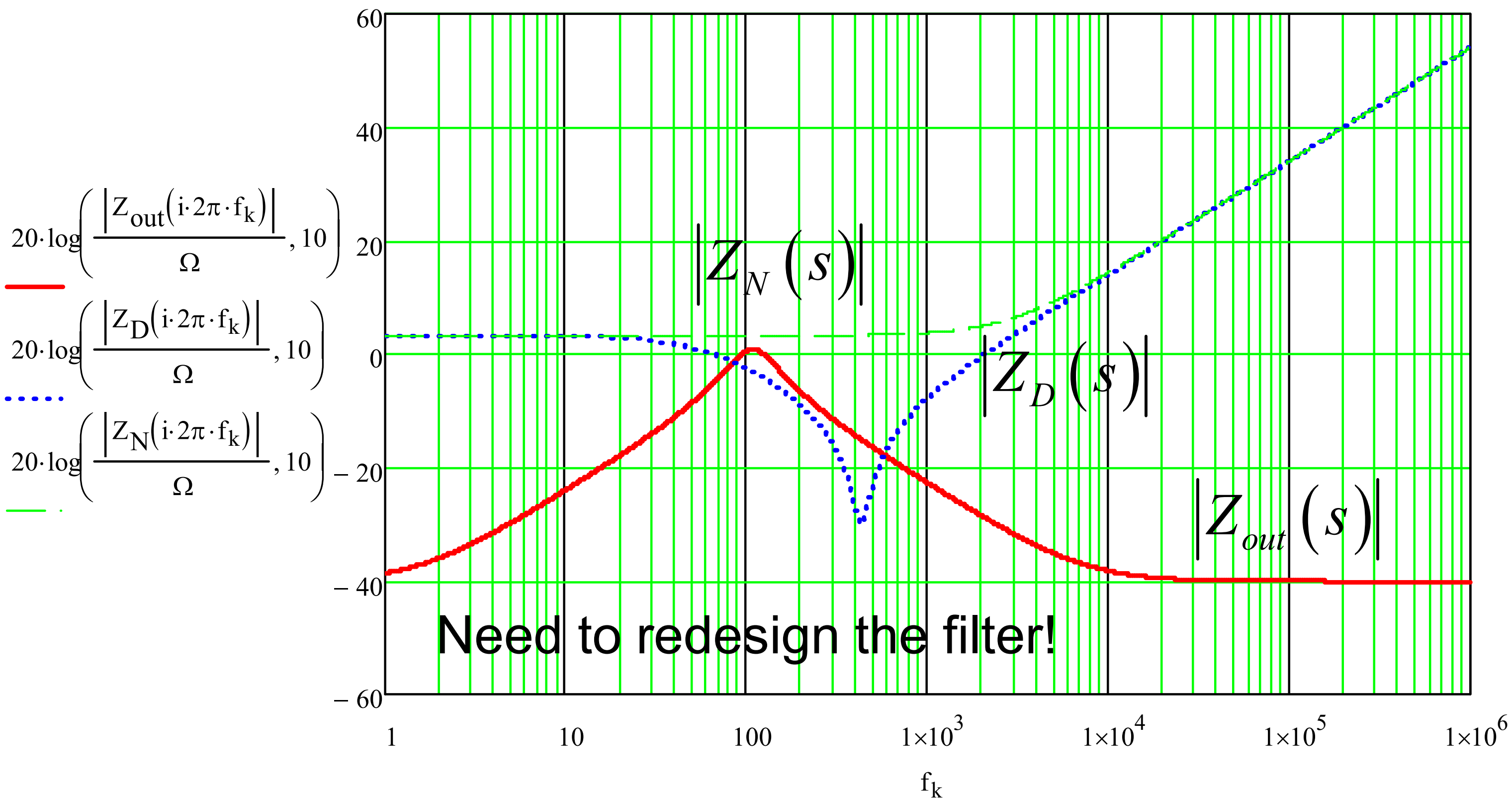
$$Z_D(s) = D'^2 R\left(\frac{1 + s\frac{L}{D'^2 R} + s^2\frac{LC}{D'^2}}{1 + sRC}\right)$$

- The original loop gain (without filter) is untouched if

$$\frac{1 + \frac{Z_{out}(s)}{Z_N(s)}}{1 + \frac{Z_{out}(s)}{Z_D(s)}} \approx 1 \quad \Rightarrow \quad \begin{array}{l} |Z_{out}(s)| << |Z_N(s)| \\ |Z_{out}(s)| << |Z_D(s)| \end{array}$$

Checking for Stability (1)

- ❑ You design the filter together with the converter
- ❑ Plot $|Z_{out}|$, $|Z_N|$ and $|Z_D|$ in the same graph
- ❑ Check that no overlap exists. If so, damp filter



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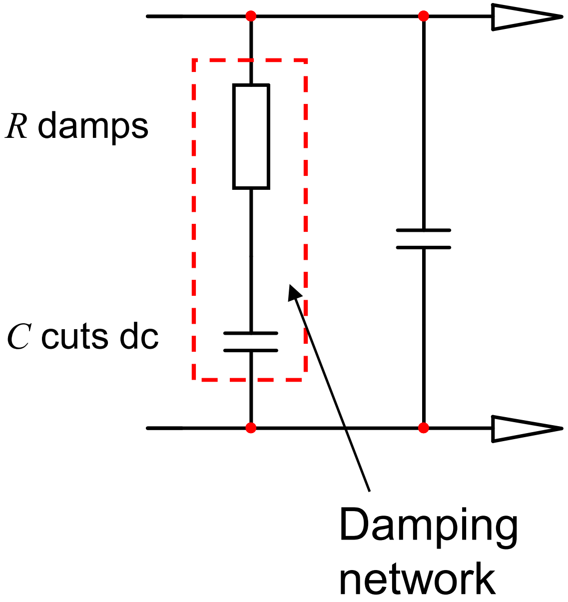
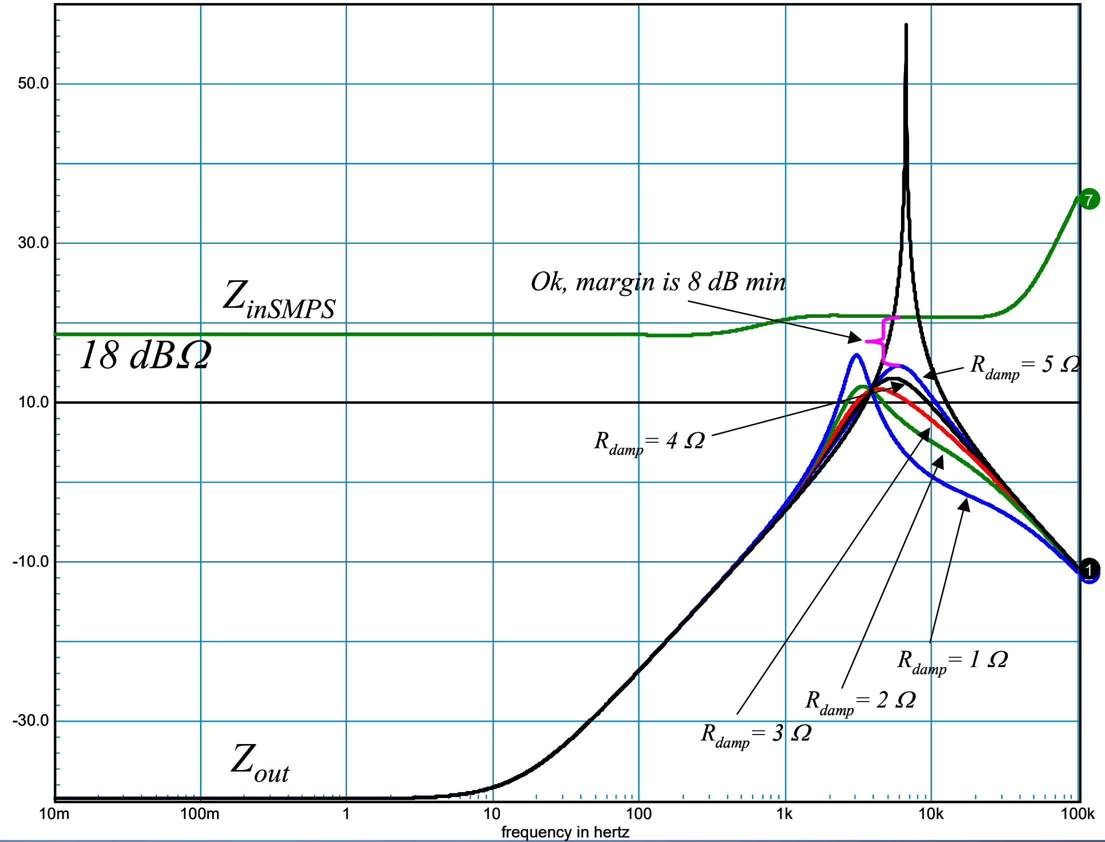
Checking for Stability (2)

- ❑ You design the filter after the converter
- ❑ Plot $|Z_{out}|$, $|Z_{in}|$ in the same graph
- ❑ Check that no overlap exists. If so, damp filter



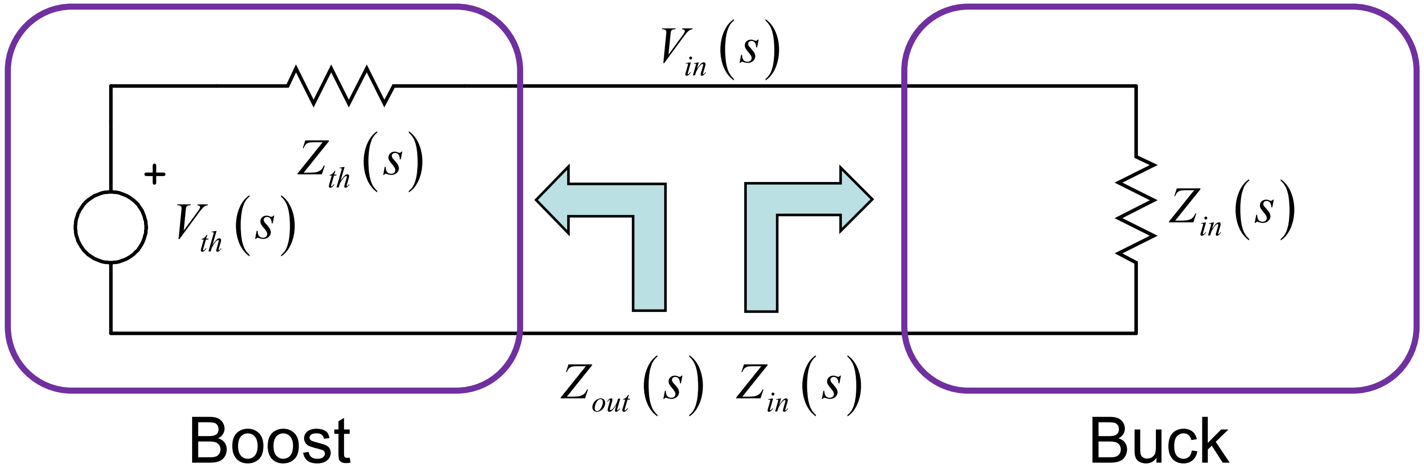


Course Agenda

- ❑ The PWW Switch Concept
- ❑ Small-Analysis in Continuous Conduction Mode
- ❑ Small-Signal Response in Discontinuous Mode
- ❑ EMI Filter Output Impedance
- ❑ **Cascaded Converters Operation**

Cascading Converters

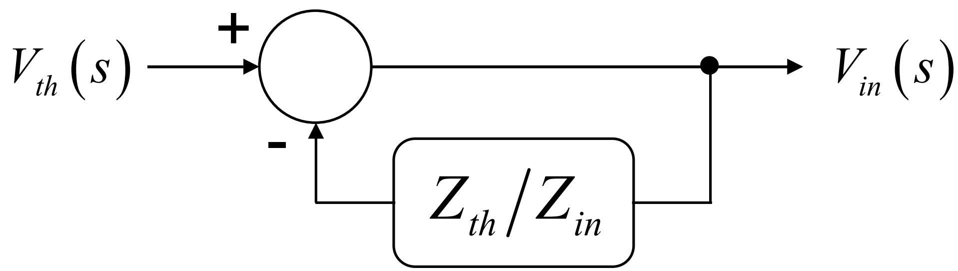
- When cascading converters, impedances also matter



- The system can be modeled by a gain T_M

$$V_{in}(s) = V_{th}(s)\frac{1}{1+\frac{Z_{th}(s)}{Z_{in}(s)}}$$

$\frac{Z_{th}(s)}{Z_{in}(s)} \rightarrow T_M(s)$



Check the Stability of the Minor Loop (1)

- The stability test requires the comparison of Z_{out} and Z_{in}
- The buck converter closed-loop input impedance is

$$\frac{1}{Z_{in}(s)} = \frac{1}{Z_N(s)}\frac{T(s)}{1+T(s)} + \frac{1}{Z_D(s)}\frac{1}{1+T(s)} \xrightarrow[\left|T(s)\right|>>1]{\text{Low frequency}} Z_{in}(s) \approx Z_N(s)$$

$$T(s) = H(s)G(s)$$

- Z_N and Z_D are defined as

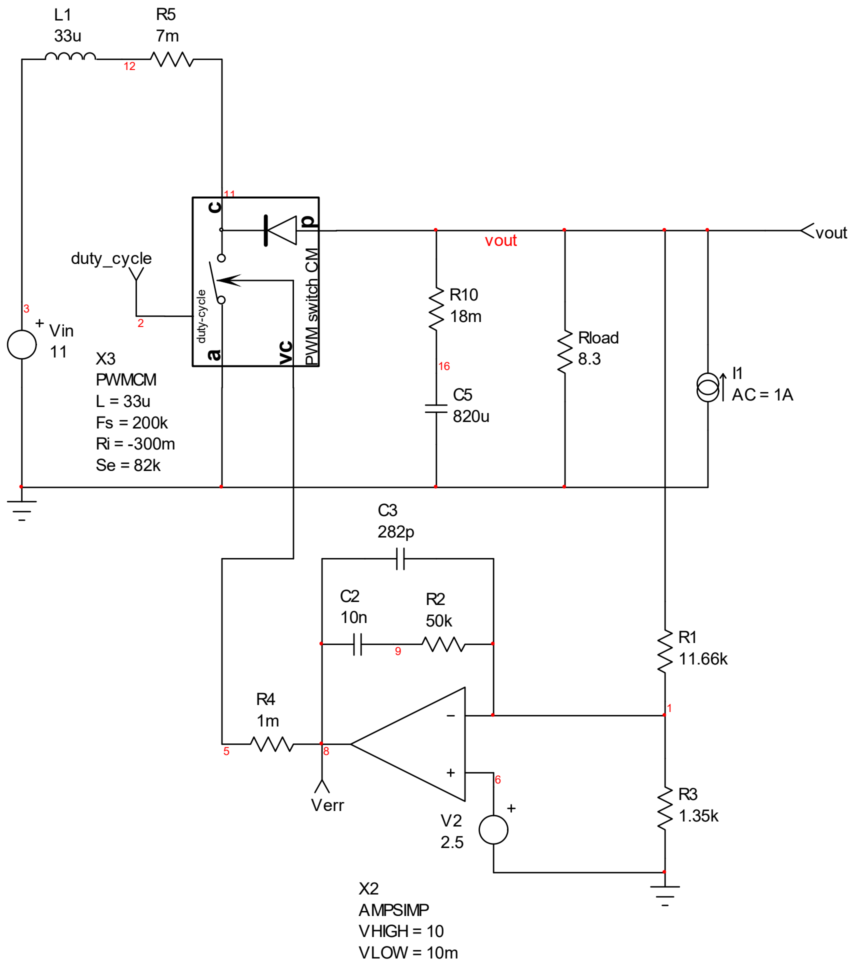
$$Z_D(s) = Z_i(s)\Big|_{D(s)=0}$$ → Open-loop input impedance

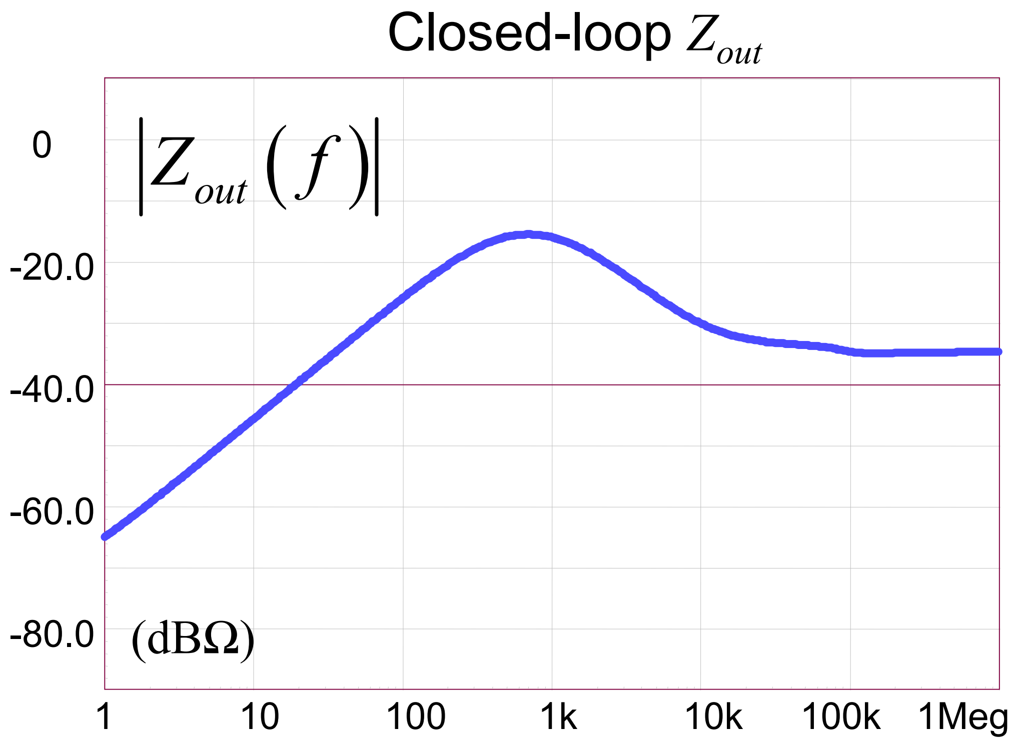
$$Z_N(s) = Z_i(s)\Big|_{V_{out}(s)=0}$$ → Input impedance for $V_{out}(s) = 0$

$$Z_D(s) = \frac{R}{D^2}\frac{1+s\frac{L}{R}+s^2LC}{1+sRC} \qquad Z_N(s) = -\frac{R}{D^2}$$

Use SPICE Models First

❑ Simulate the boost converter output impedance

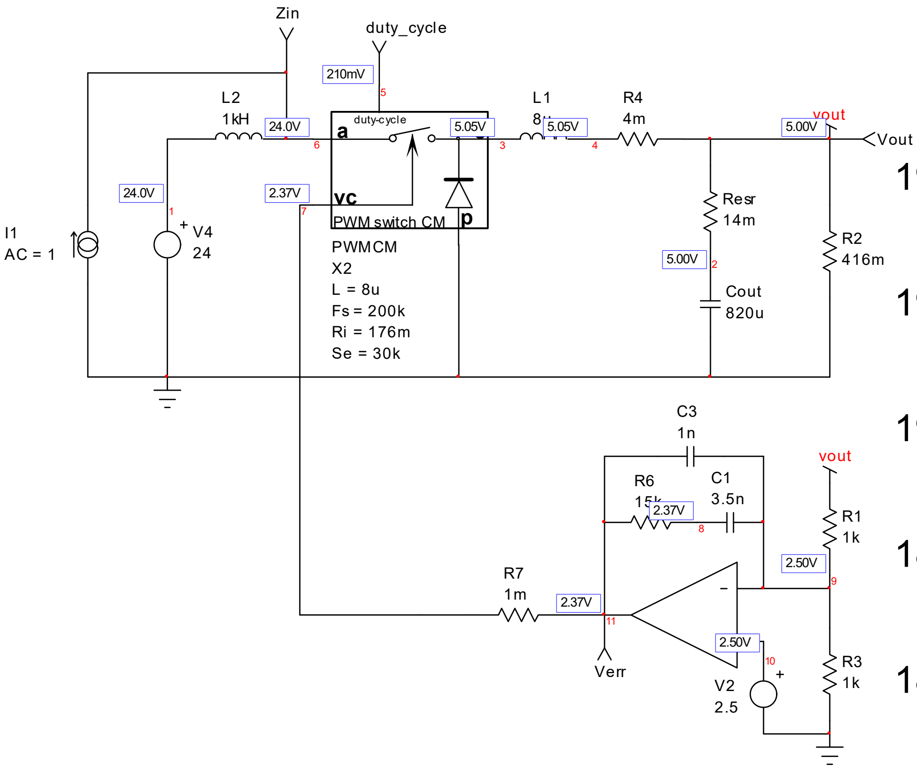


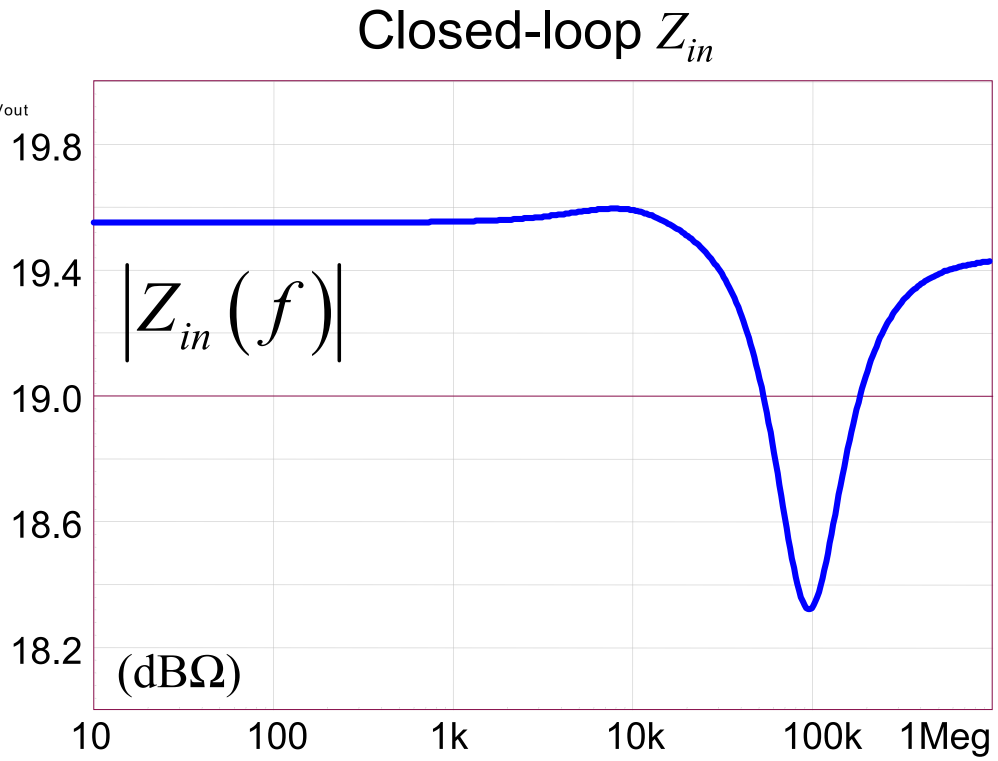


11-15 V/24 V – 3 A

Use SPICE Models First

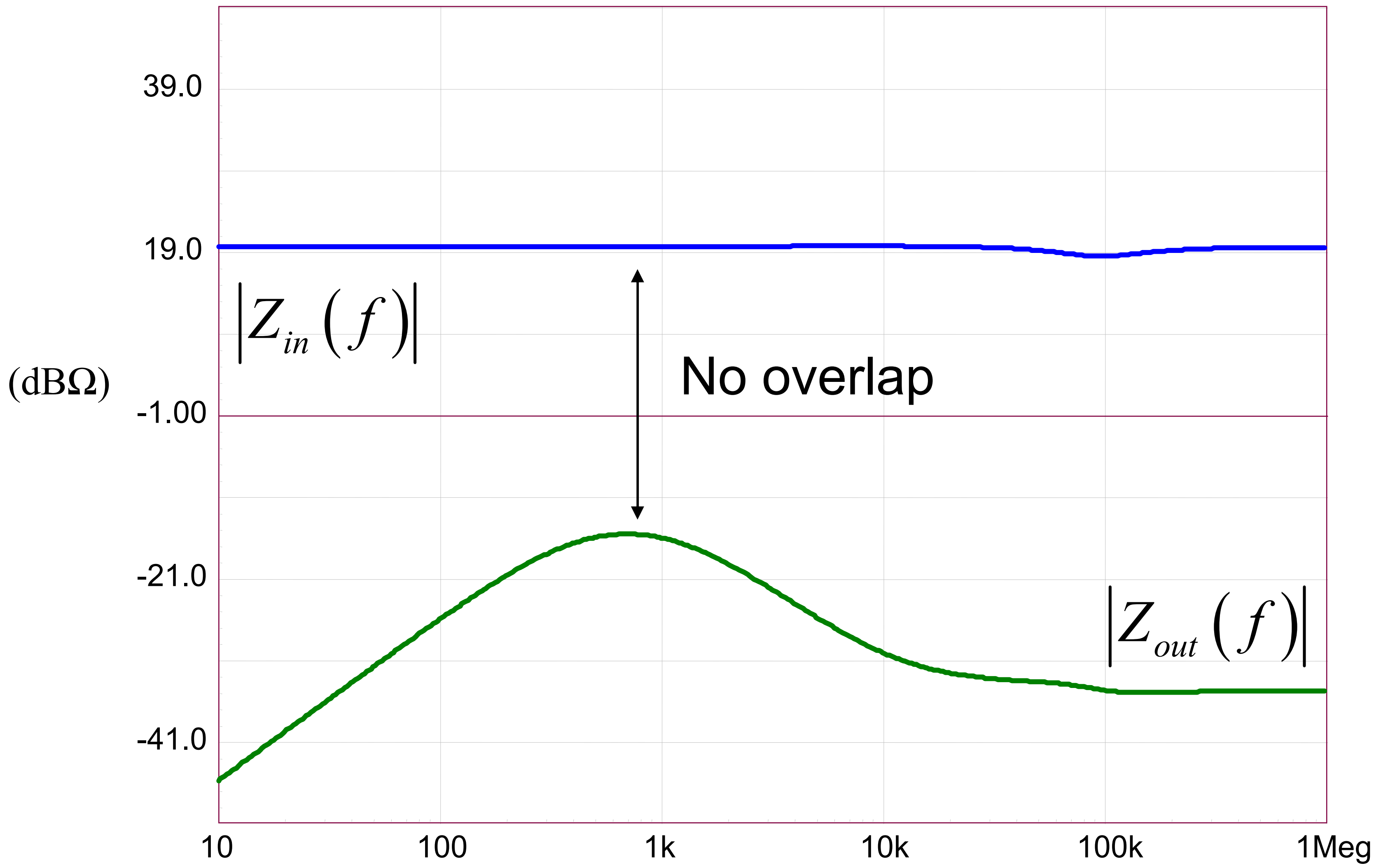
- ❑ Simulate the buck converter input impedance



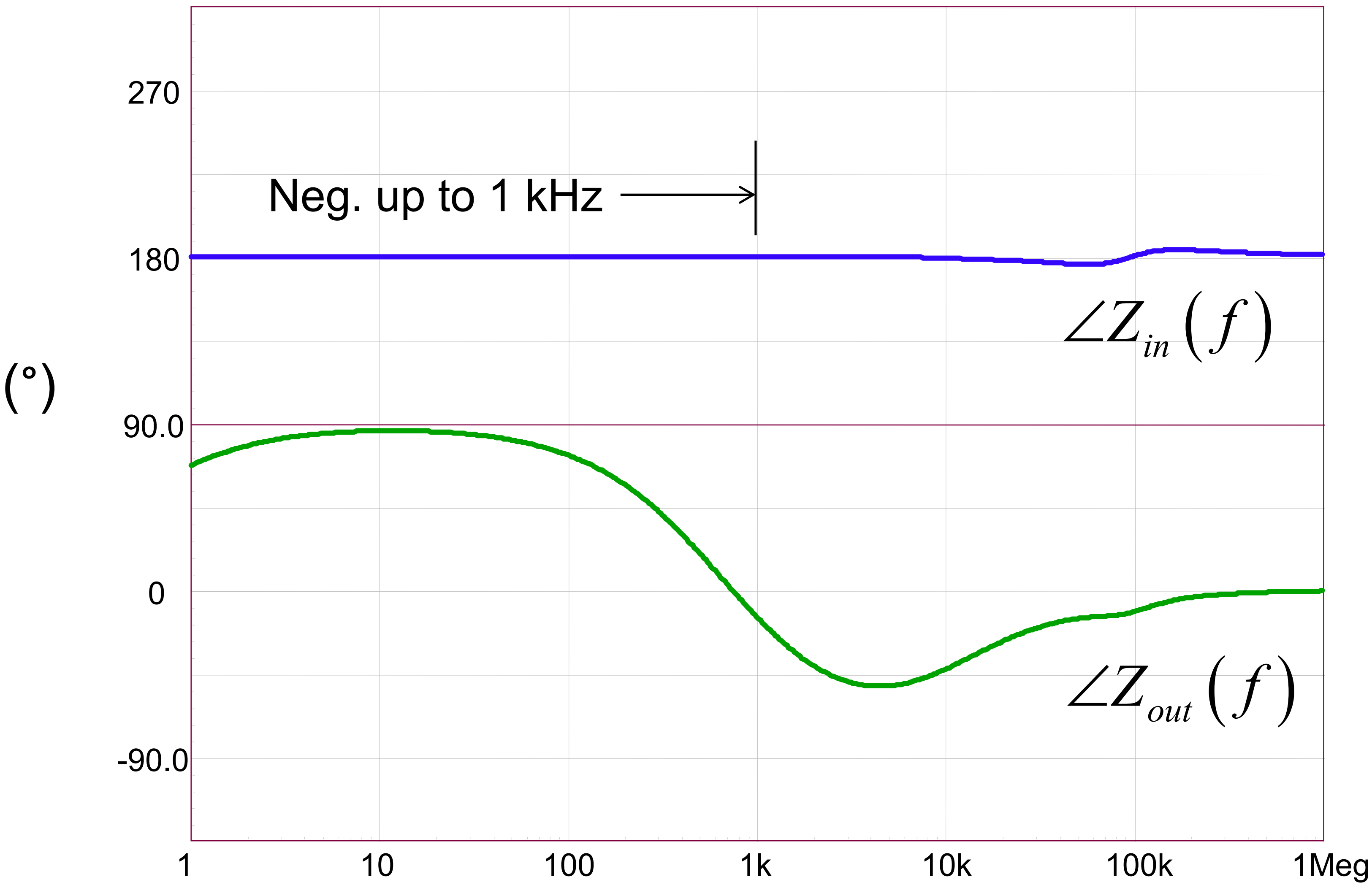


24 V/5 V – 12 A

Compare Magnitude Curves

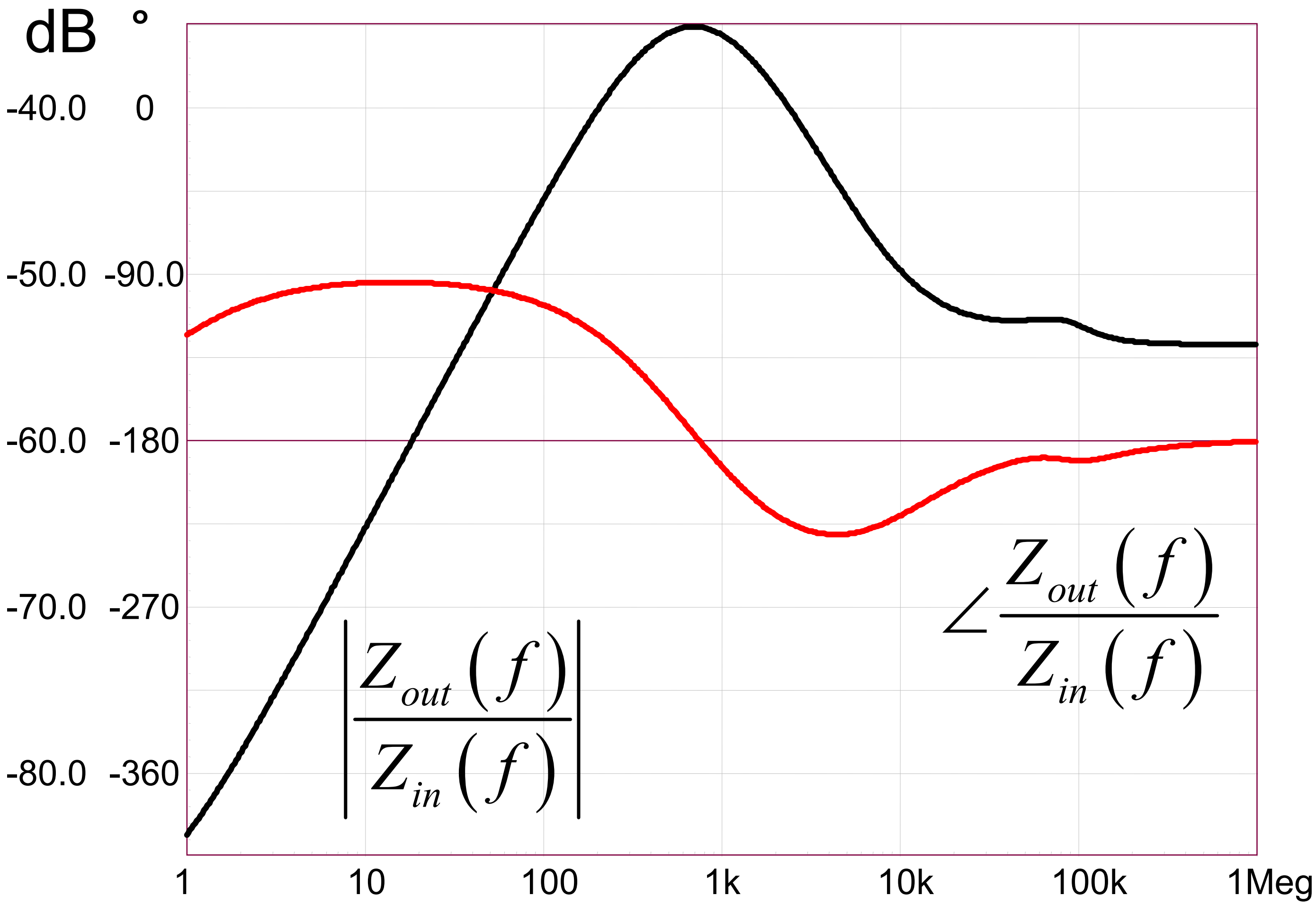


Compare Phase Curves

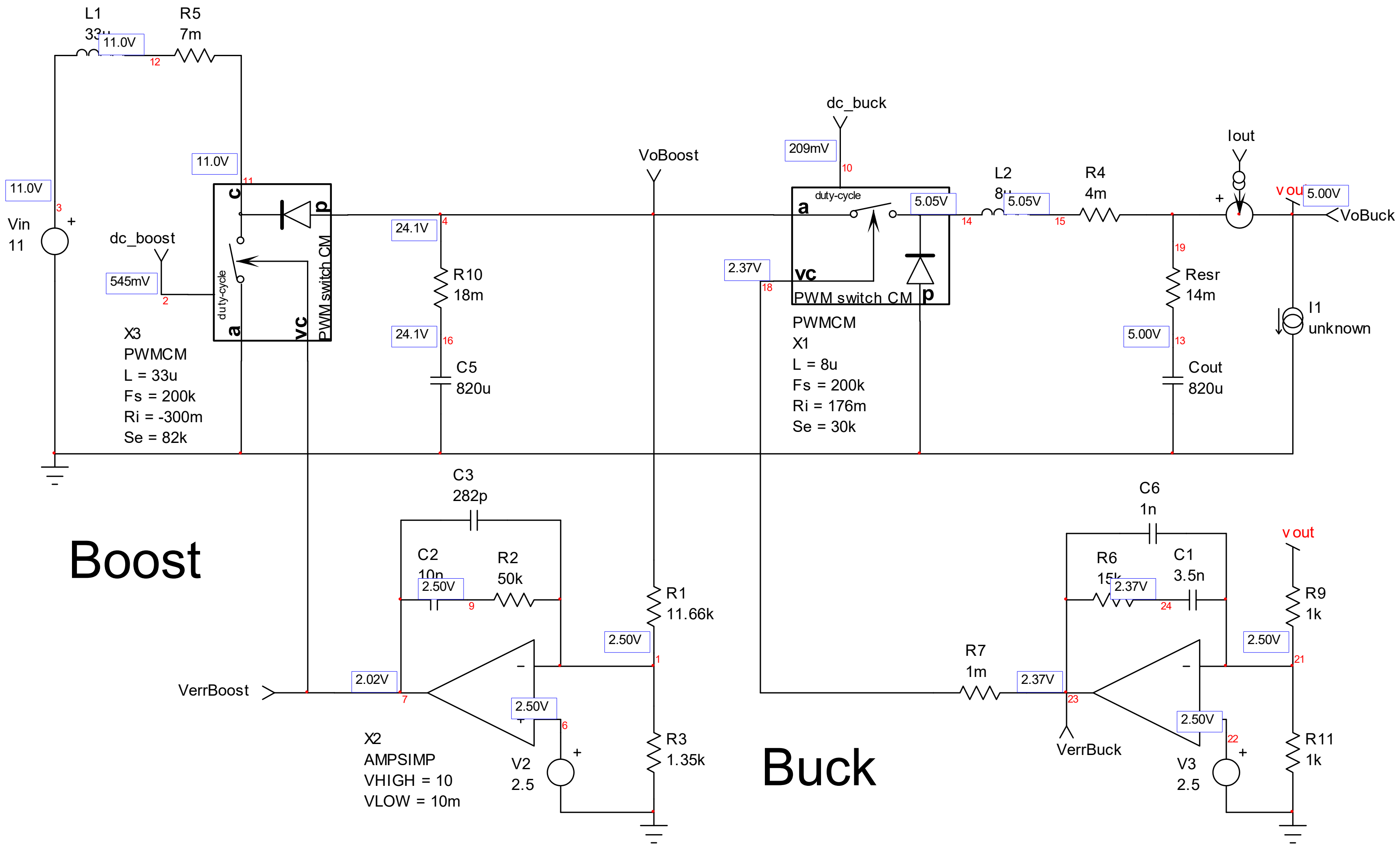


Check Minor Loop Gain

- There is no gain, $\left|Z_{out}\right| << \left|Z_{in}\right|$. The system is stable



With Cascaded Converters



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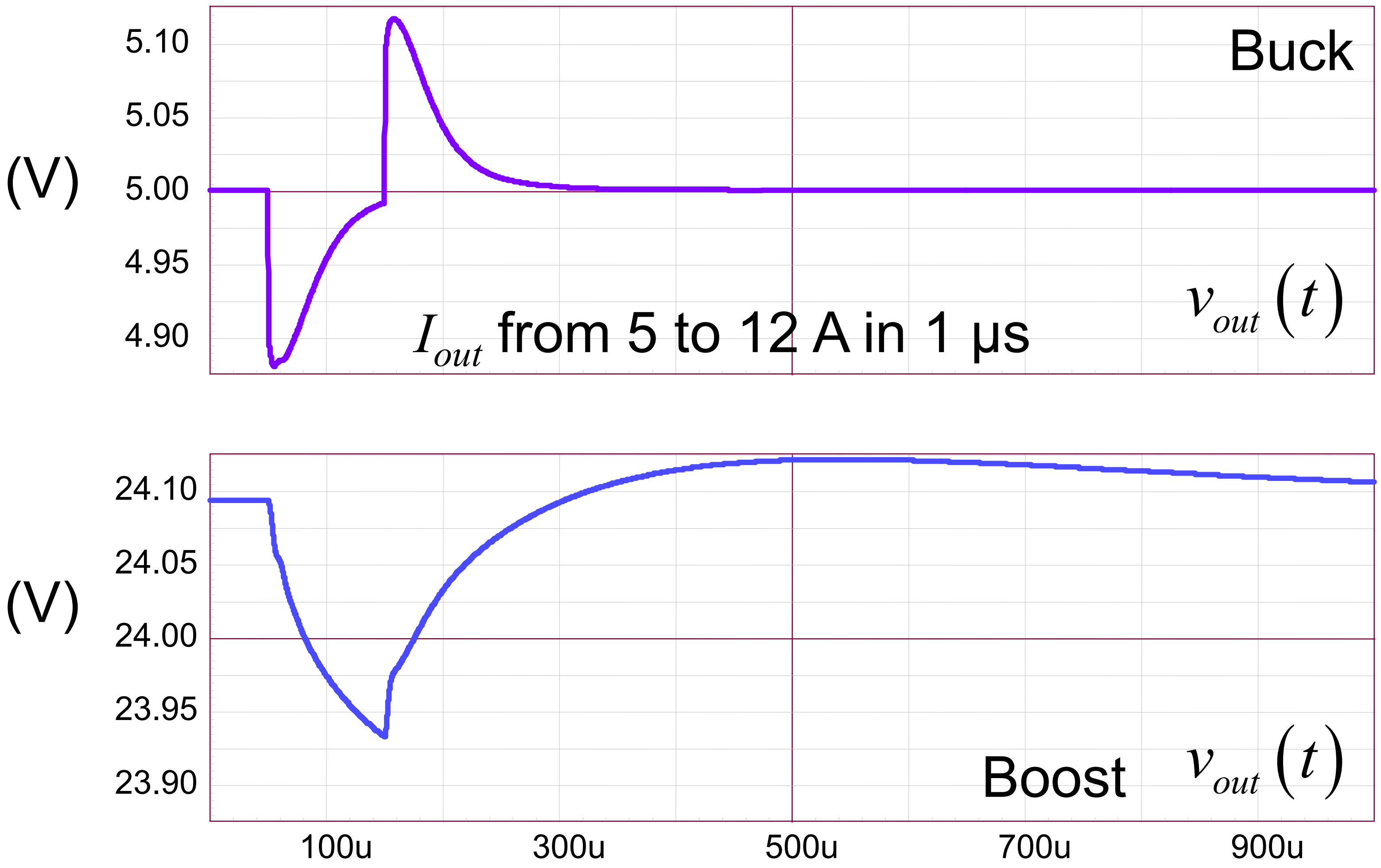
Check Transient Response

- ❑ Transient response is good for the buck



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“Designing Control Loops for Linear and Switching Power Supplies”, C. Basso

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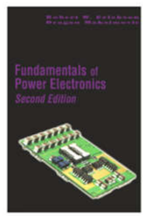

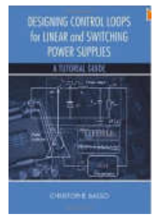

Conclusion

- The PWM switch model is an essential tool for modeling
- It can be used to derive small-signal response of a boost in CM
- Its SPICE implementation helps test various transfer functions
- It can be used to assess open- and closed-loop parameters
- Interactions with the EMI input filter can be analyzed
- Cascaded converters stability can be checked with the model
- Practical measurements on the bench must always be performed