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Small-Signal Modeling and Analytical Analysis of Power Converters

Christophe Basso

IEEE Senior Member

Course Agenda

- ❑ Linear and Non-Linear Functions
- ❑ What is a Small-Signal Model?
- ❑ Fast Analytical Techniques at Work
- ❑ From a Switched to Linearized Model
- ❑ The CCM VM Small-Signal PWM Switch Model
- ❑ The DCM VM Small-Signal PWM Switch Model
- ❑ Peak Current Mode Control in Large Signal
- ❑ The CCM CM Small-Signal PWM Switch Model
- ❑ The DCM CM Small-Signal PWM Switch Model
- ❑ The PWM Switch in Boundary Mode

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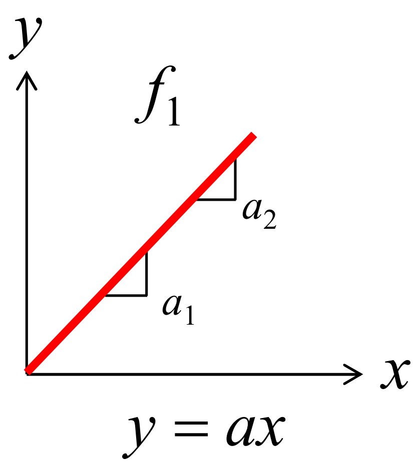
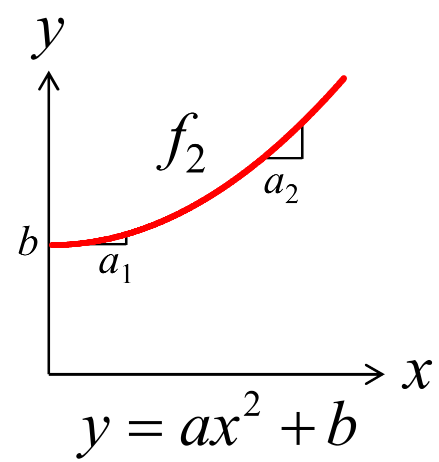
Linear and Nonlinear Functions

- Functions or systems can be linear or non-linear





- The slope is constant along the considered section:

$a_1 = a_2$ ⇒ the function f_1 is linear

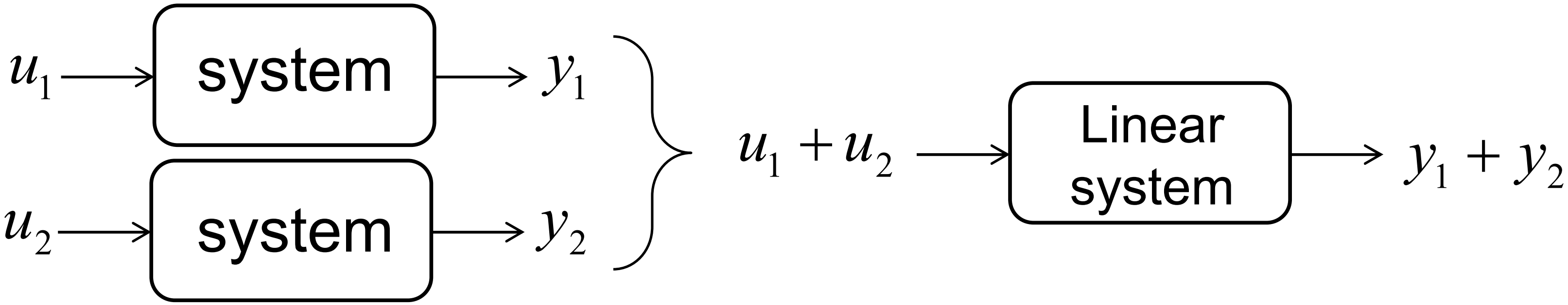
- The slope is not constant along the considered section:

$a_1 \neq a_2$ ⇒ the function f_2 is non linear

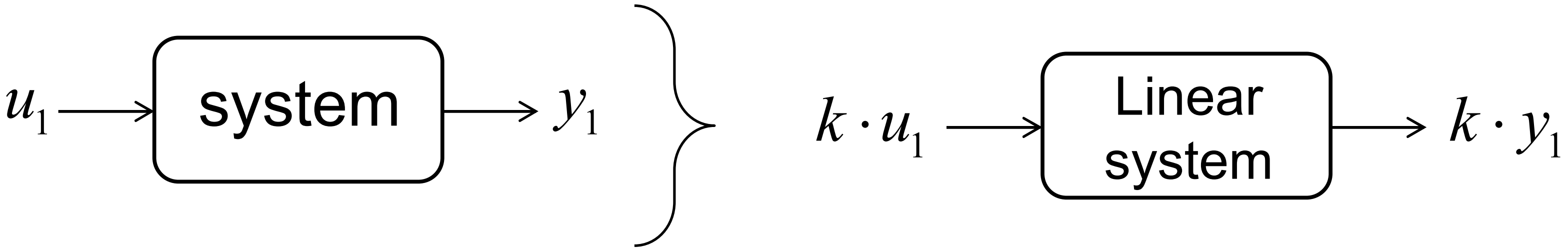
The Definition of Linear

- A linear system must fulfill the superposition principle

$$f(x+y) = f(x) + f(y)$$ Additivity

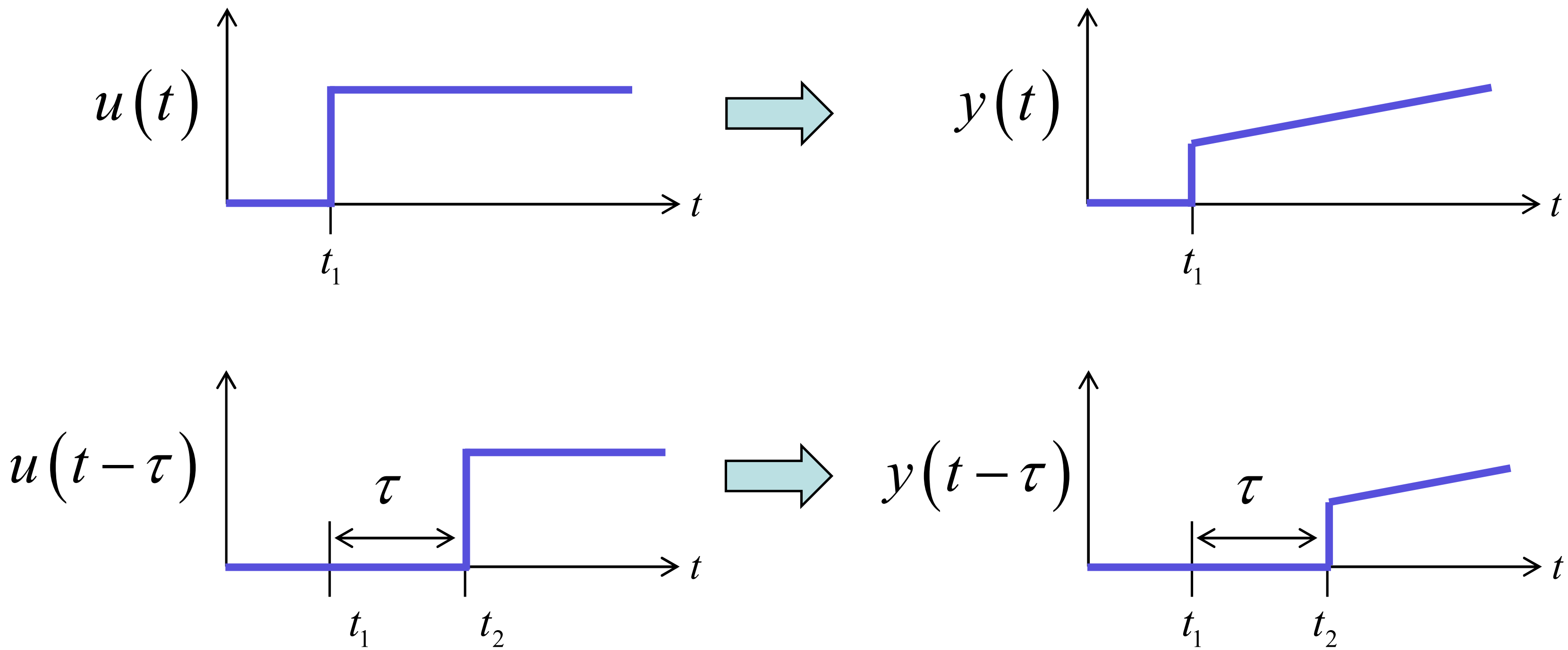


$$f(k \cdot x) = k \cdot f(x)$$ Proportionality of homogeneity



Time Invariance

- System properties do not change as time elapses



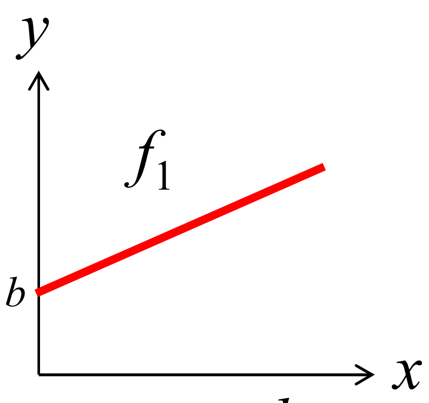
Superposition

Time invariance

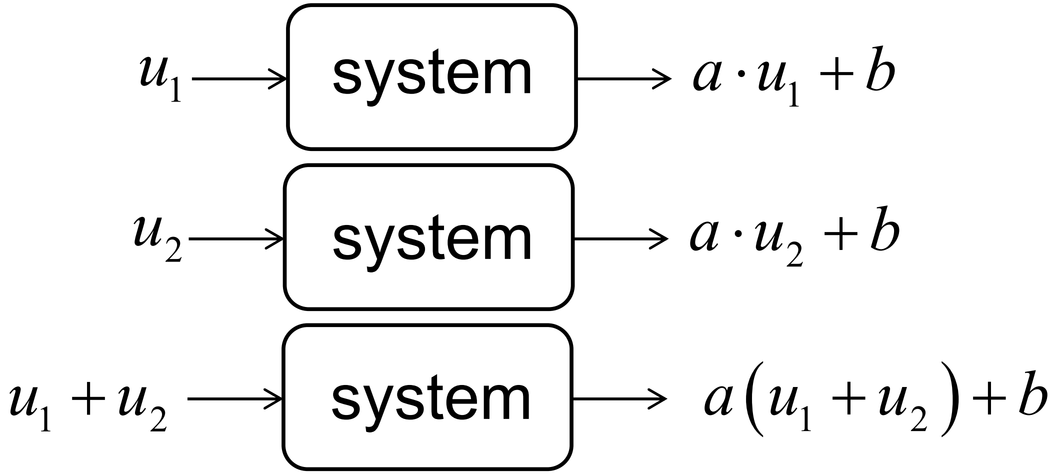
→ Linear Time Invariant (LTI)

Who is Linear?

- Check properties of the linear function with an offset

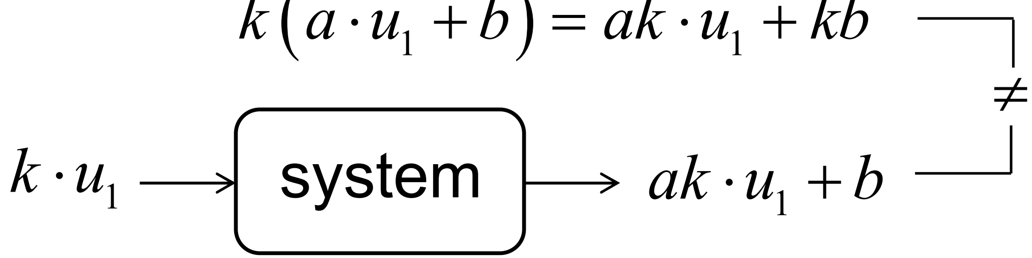


$y = ax + b$



$$a(u_1 + u_2) + b \neq a(u_1 + u_2) + 2b$$

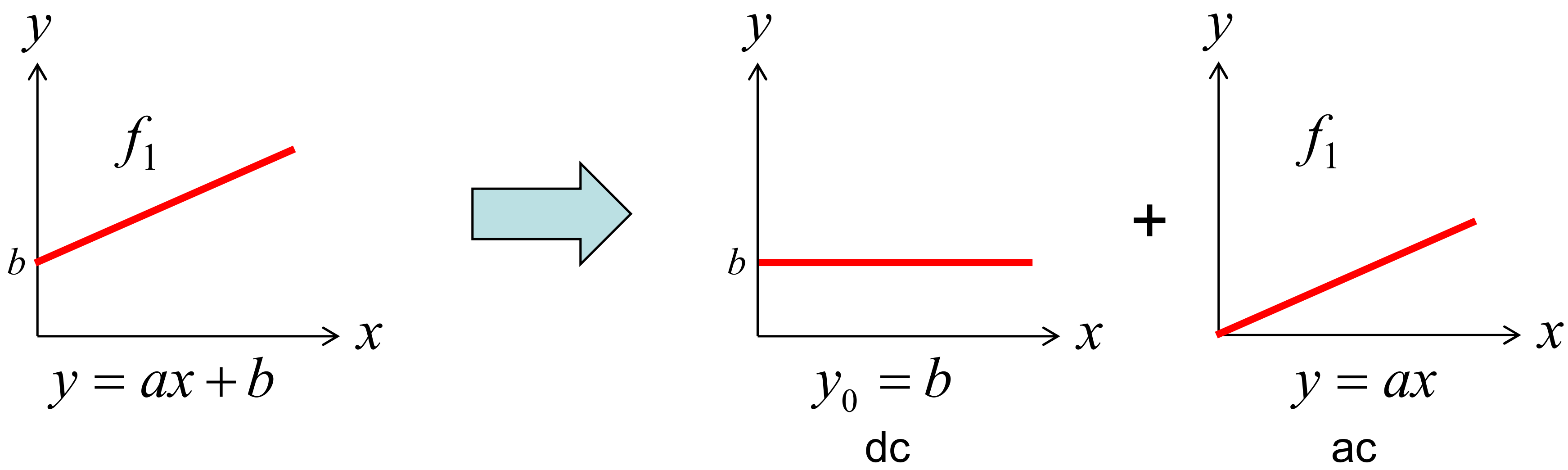
$$k(a \cdot u_1 + b) = ak \cdot u_1 + kb$$



Nonlinear!

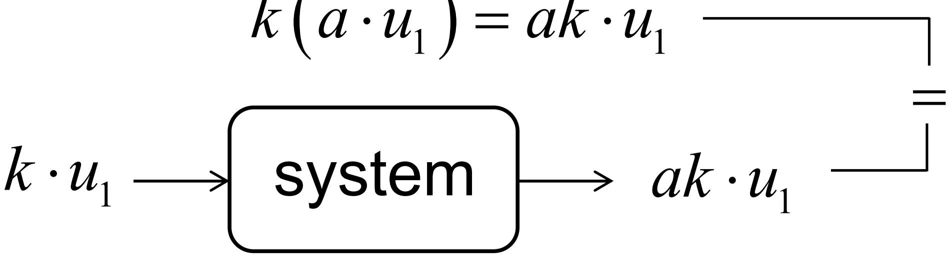
How to Linearize?

- ❑ Split the function in two parts, offset (dc) and curve (ac)



$$a\left(u_1 + u_2\right) = a \cdot u_1 + a \cdot u_2$$

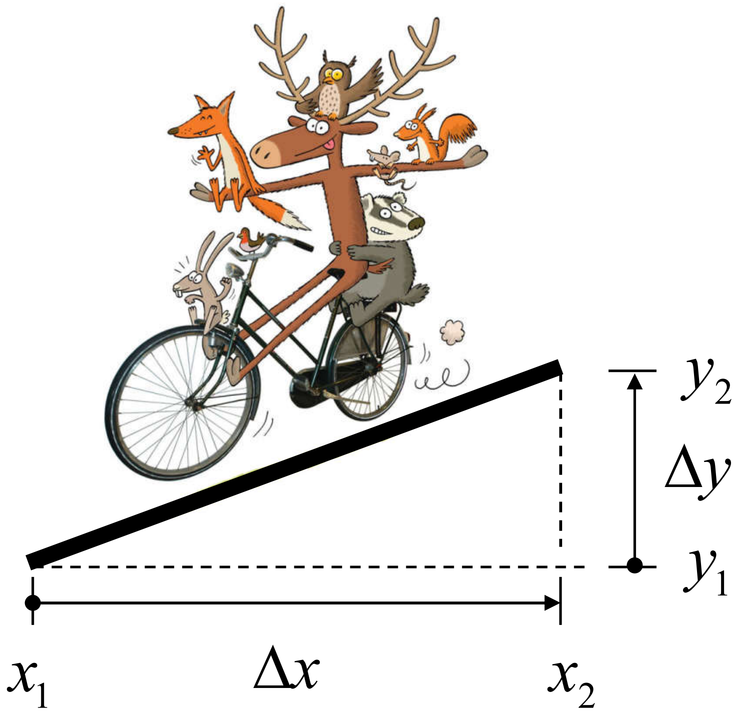
$$k\left(a \cdot u_1\right) = ak \cdot u_1$$



$y = ax$ is linear!

How Do You Calculate The Slope?

- Look at the vertical variation brought by a horizontal step



$$\Delta x = x_2 - x_1$$

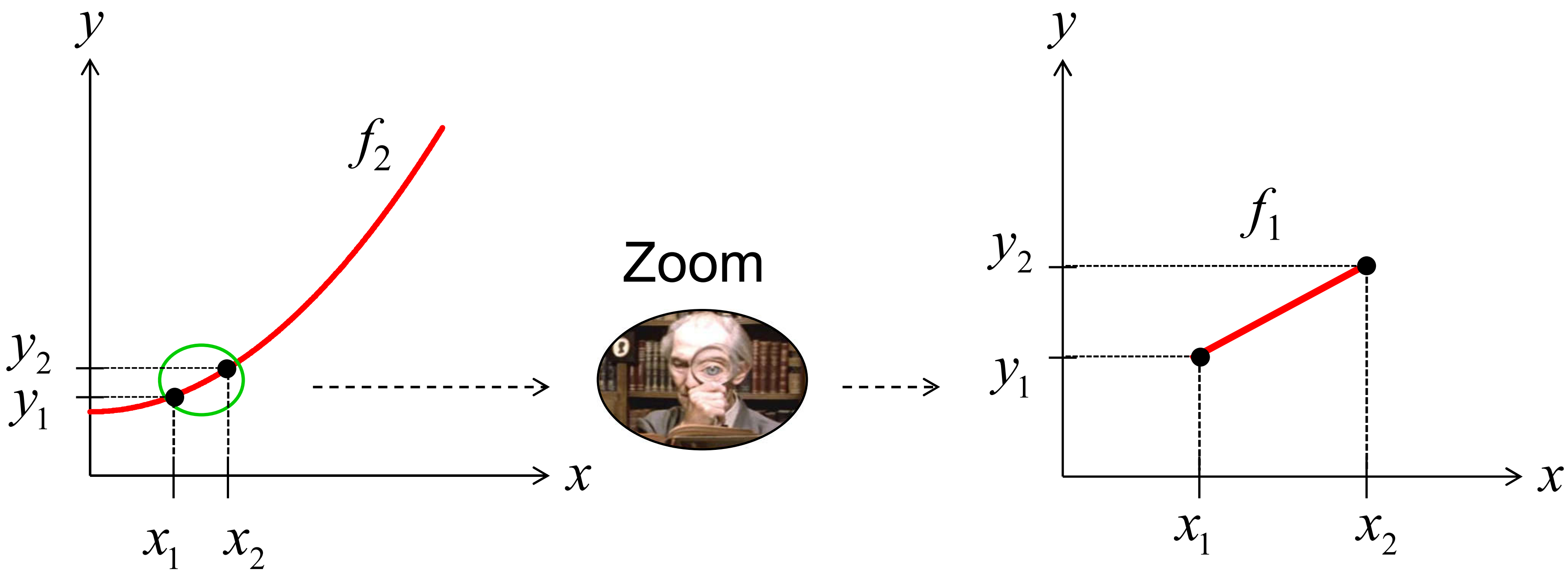
$$\Delta y = y_2 - y_1$$

$$a = \frac{\Delta y}{\Delta x}$$

- As y_2 is larger than y_1, the slope in this case is positive

Linearity in a Non-Linear Function

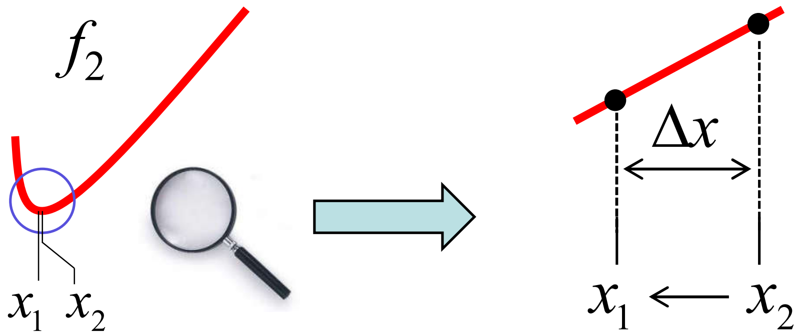
- A non-linear function can be linear at an observation point



- x_2 - x_1 must be small enough to isolate a linear section
- Zooming further means having x_2 x_1 close to each other

From Slope to Differentiation

- To see a linear zone Δy and Δx must be very small quantities



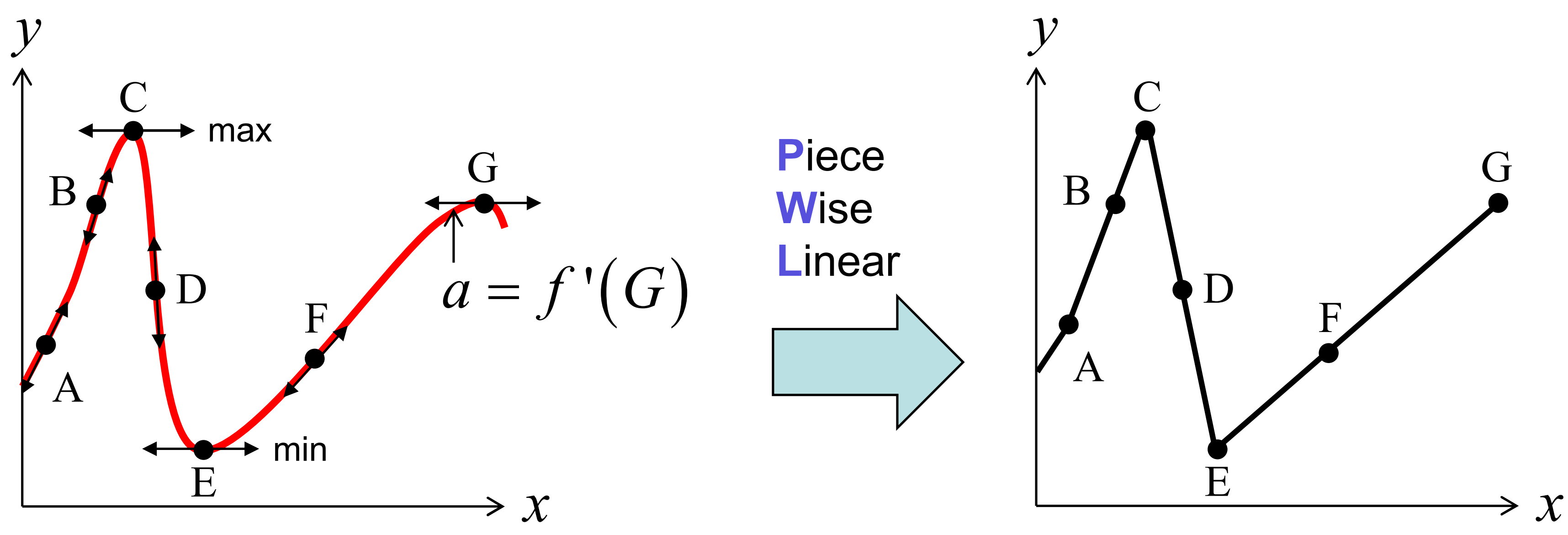
- When x_2 is close to x_1, we have calculated the slope at x_1
- The slope at this point is called the derivative of f at x_1

$$f'(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1+\Delta x) - f(x_1)}{(x_1+\Delta x) - x_1} \quad \xrightarrow{\text{Also noted}} \quad \frac{df(x)}{dx}$$

- Infinitesimal variations are noted dy and dx (Leibniz)

Tangent and Differentiation

- A tangent is a graphical representation of a differentiation

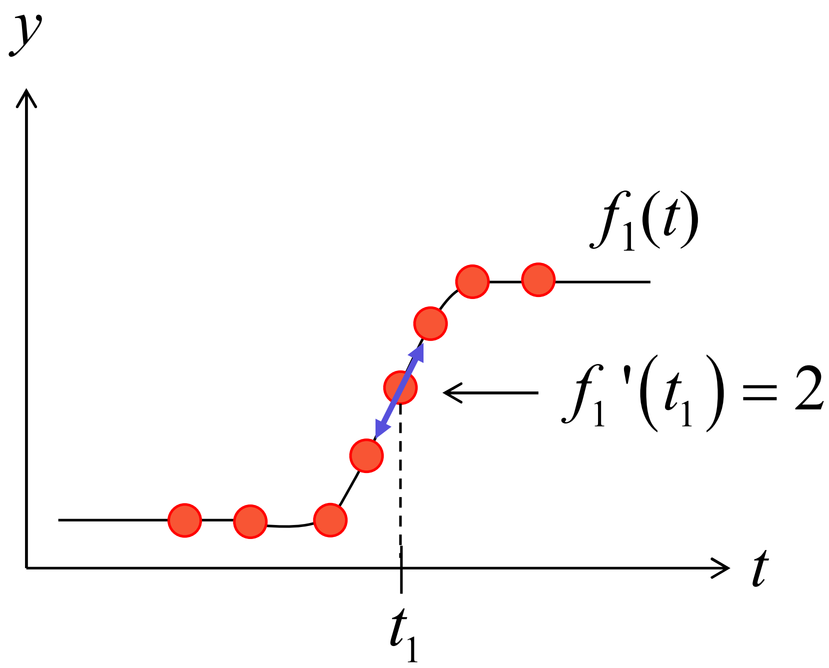


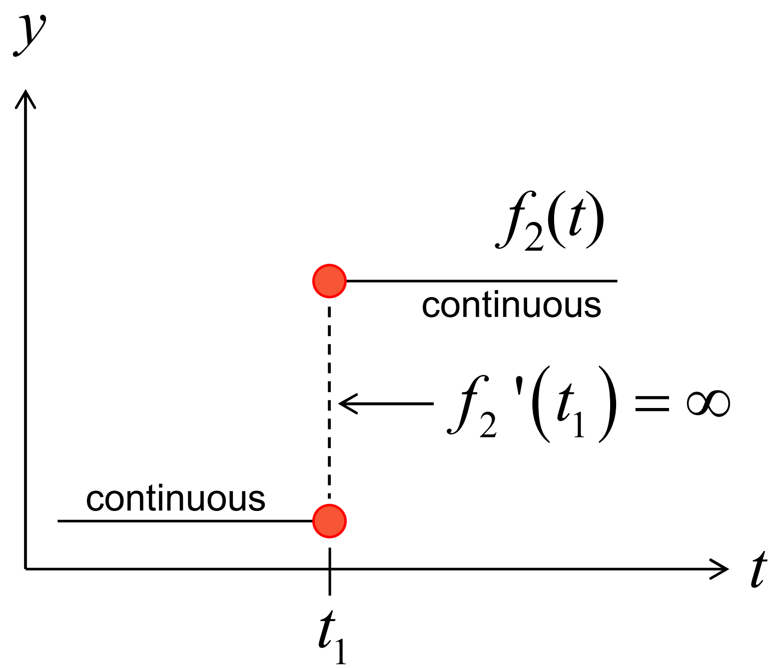
- When the slope is zero, the function is minimum or maximum

- A PWL is a linear approximation of a non-linear function

Differentiation and Continuity

- "A small input change gives a small output change"





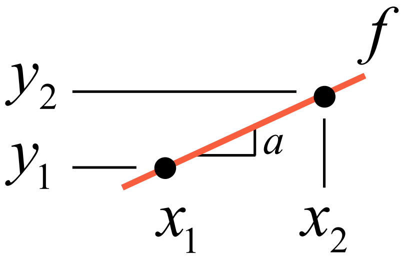
$$\lim_{t \to t_1} f_1(t) = f_1(t_1)$$

$$\lim_{t \to t_1} f_2(t) = ?$$

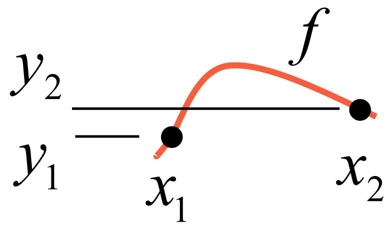
- Function f_2 is discontinuous at point t_1: no finite differentiation

Why Do We Need Linear Equations?

- ☐ If the function is linear, we can extrapolate points positions

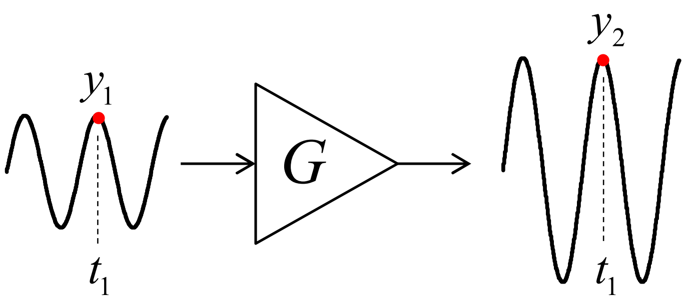


$$y_2 = y_1 + a\left(x_2 - x_1\right)$$

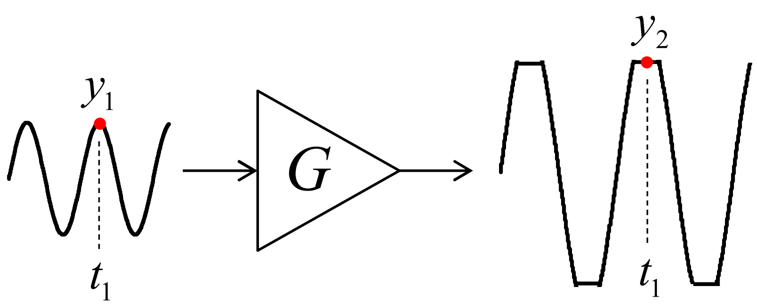


$$y_2 = ?$$

- ☐ Linearity implies no gain change during modulation



$$y_2 = y_1 G$$

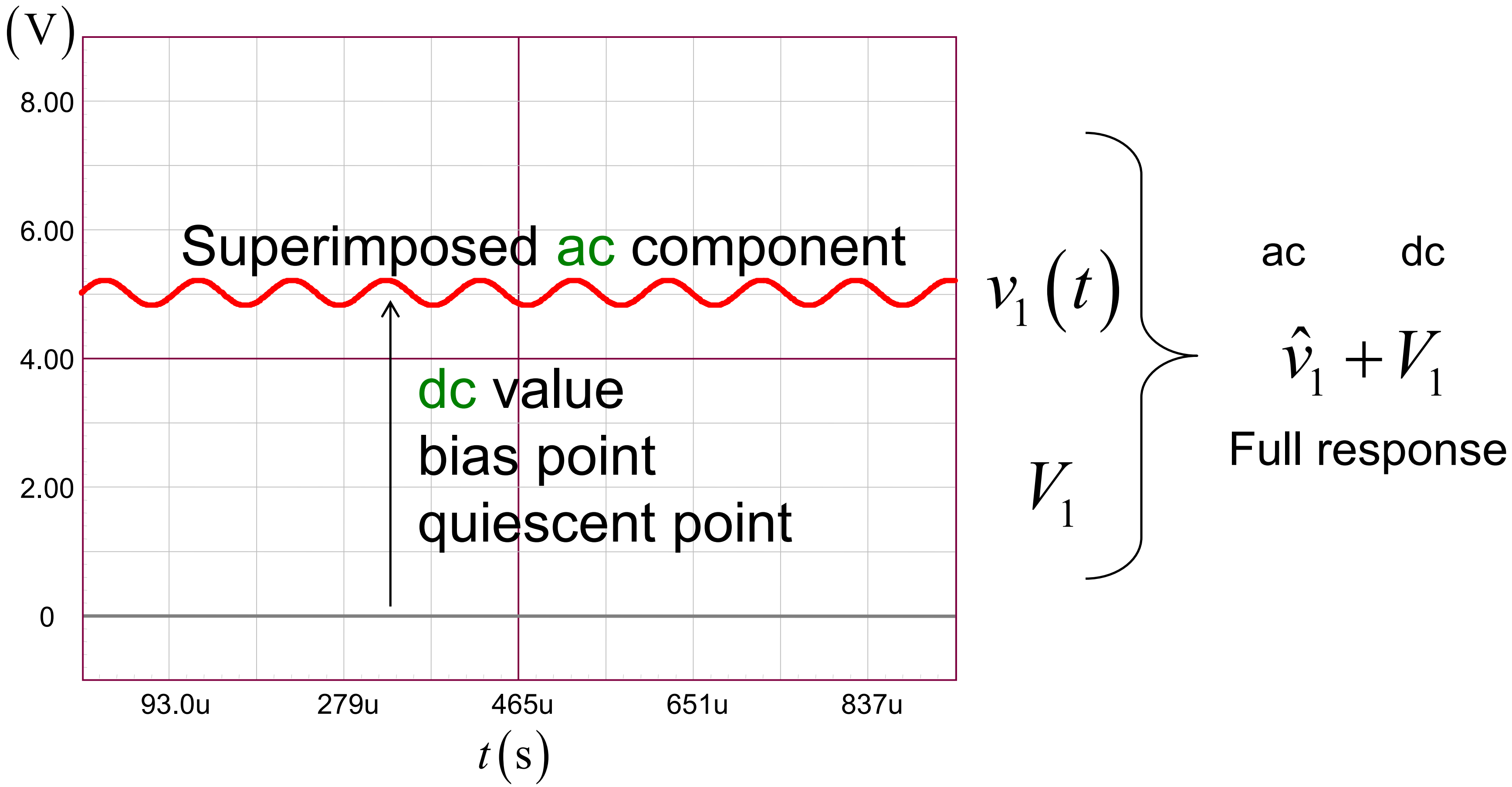


$$y_2 = ?$$

- ☐ Frequency response exploration requires linear blocks

What Does Small-Signal Operation Mean?

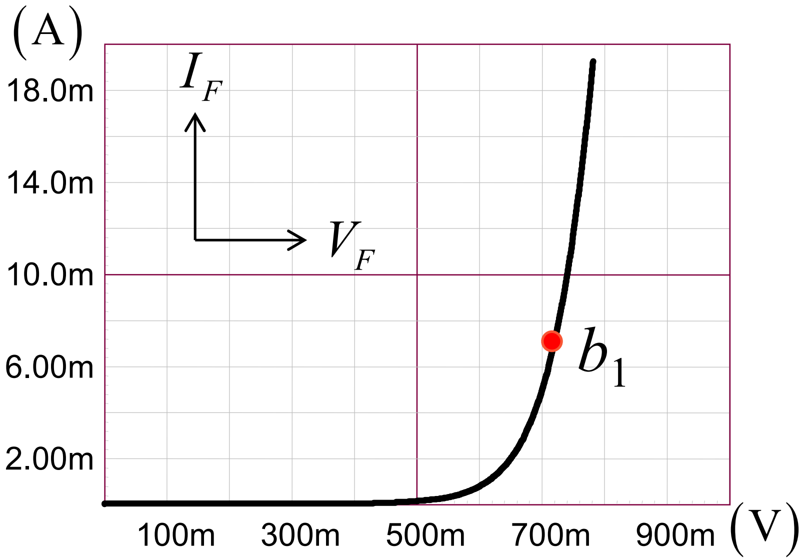
- A small amplitude ac component straddles a dc level

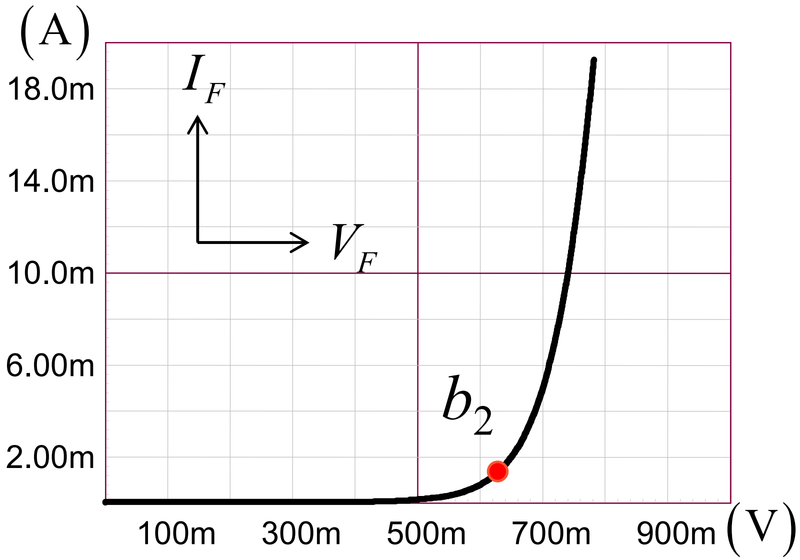


- The signal amplitude is small to maintain linearity

Bias Point and Linear Zone

- ❑ Where does the device operate in its characteristics?





$I_1 = 7\,\text{mA} \quad V_1 = 710\,\text{mV}$ $\qquad$ $I_1 = 1.8\,\text{mA} \quad V_1 = 620\,\text{mV}$

- ❑ Bias point tells where the device stands in its dc response
- ❑ Please note capital letters for dc values

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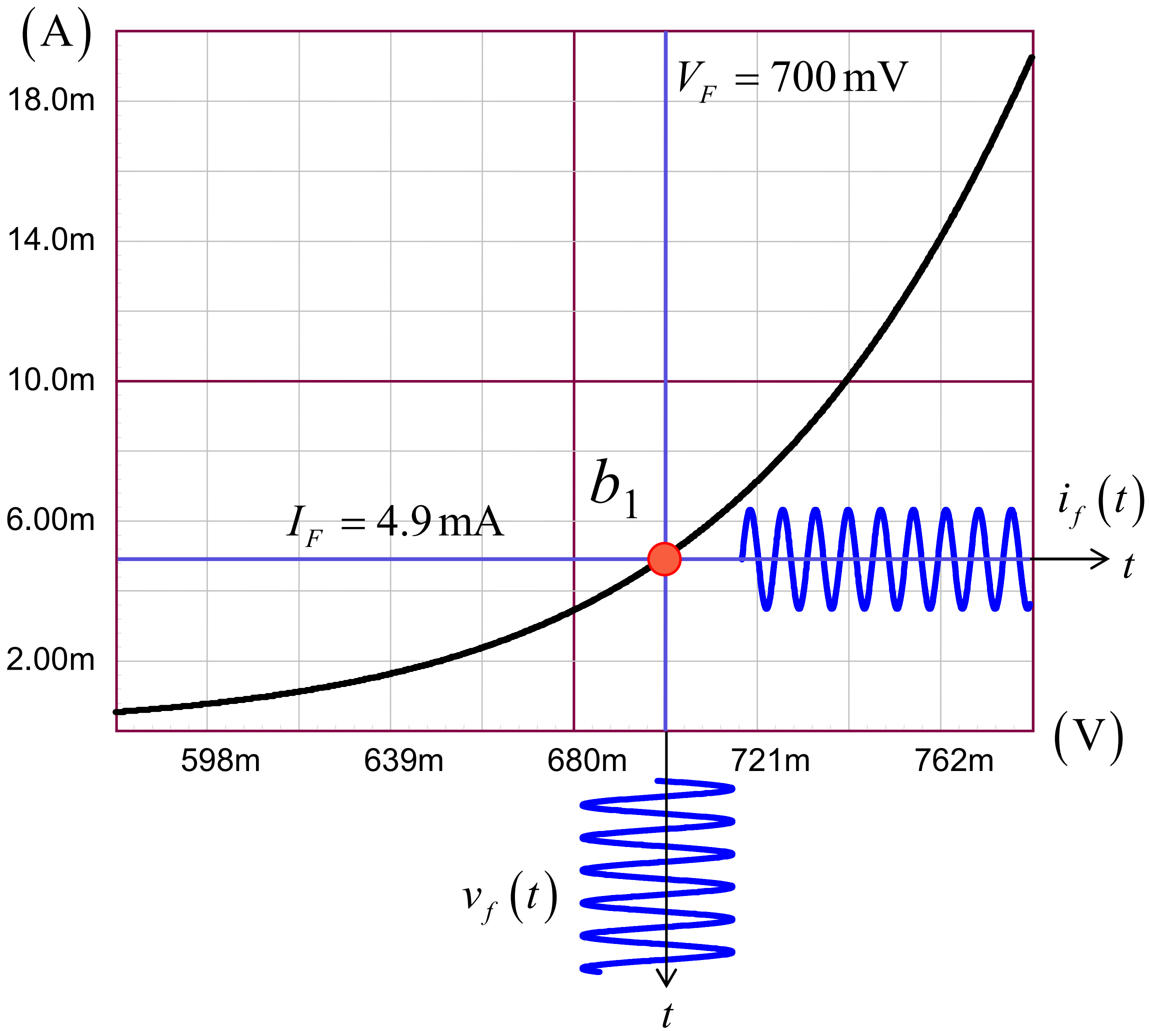
Ac Modulation at Bias Point 1

- ❑ The ac modulation linearly moves the bias point

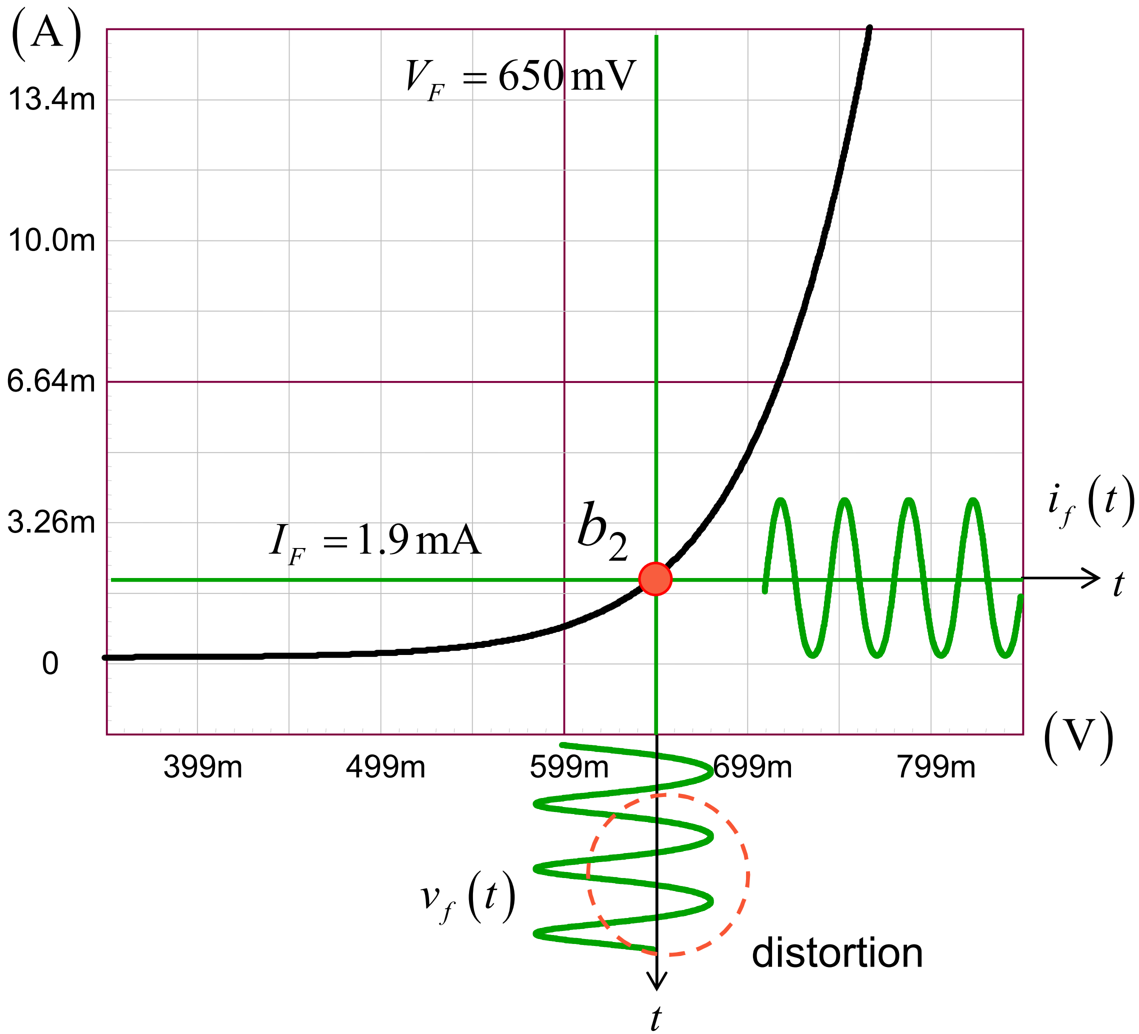


The modulation amplitude is small enough: system is in linear mode

Small-signal excitation

Ac Modulation at Bias Point 2

- Bias point change brings distortion to the modulated signals



The modulation amplitude is too big: system is in nonlinear mode

Large-signal excitation

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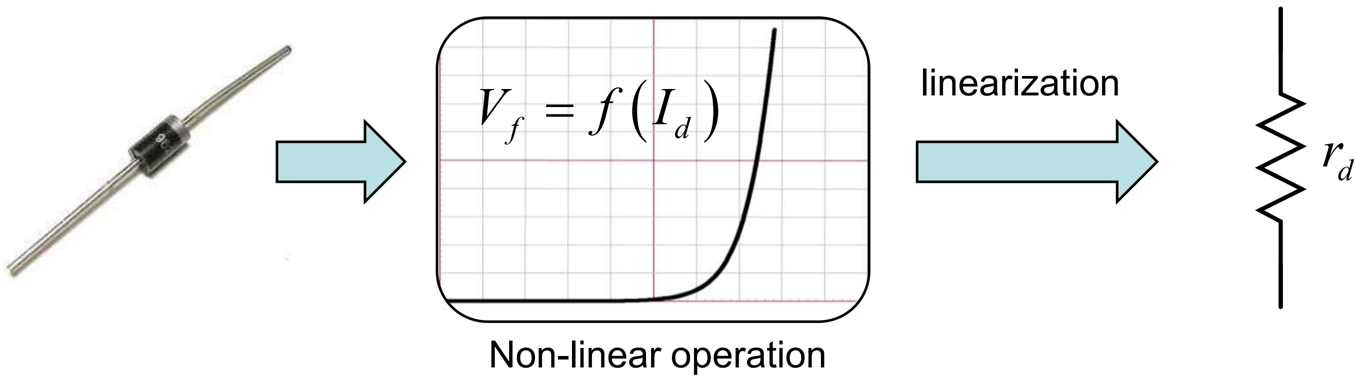
A Small-Signal Model

- ❑ How can we predict the ac response from the diode?

1. We need a model accounting for the operating point
2. It must reflect the relationship between I and V at this point.

We need a small-signal model



Methods to Derive a Small-Signal Model

- Start from the non-linear equation

1. Perturb the equation: add a small ac variation to all variables
2. Re-arrange, sort dc and ac terms

- The current in the diode obeys Shockley equation:

$$I_F = I_S\left(e^{\frac{V_F}{nV_T}} - 1\right)$$

I_F the diode forward current

V_F the diode forward drop

V_T thermal voltage, ≈26 mV at 25 °C

n emission coefficient, varies between 1 and 2

I_S the reverse bias saturation current, few nA for a 1N4148

Method 1, Equation is Perturbed

- Considering the exponential much larger than 1, we have:

$$I_F = I_S\left(e^{\frac{V_F}{nV_T}} - 1\right) \approx I_S e^{\frac{V_F}{nV_T}}$$

- Now, each variable is perturbed by a small ac variation
- The variation is considered small, the system remains linear

$$I = f\left(V_{in}, V_{out}\right) \quad \xrightarrow{\text{perturb}} \quad I + \hat{i} = f\left(V_{in} + \hat{v}_{in}, V_{out} + \hat{v}_{out}\right)$$

- I_S, n and V_T are constant, perturb only I_F and V_F:

$$I_F + \hat{i}_F = I_S e^{\frac{V_F + \hat{v}_F}{nV_T}} = \underbrace{\boxed{I_s e^{\frac{V_F}{nV_T}}}}_{I_F} e^{\frac{\hat{v}_F}{nV_T}} = I_F e^{\frac{\hat{v}_F}{nV_T}}$$

Method 1, Equation is Perturbed

- Use the Padé approximant of order 1: $e^x \approx 1 + x$

$$\underbrace{I_F + \hat{i}_F}_{\text{Full response}} = I_F\left(1 + \frac{\hat{v}_F}{nV_T}\right) = \underbrace{I_F}_{\text{dc}} + \underbrace{I_F \frac{\hat{v}_F}{nV_T}}_{\text{ac}}$$

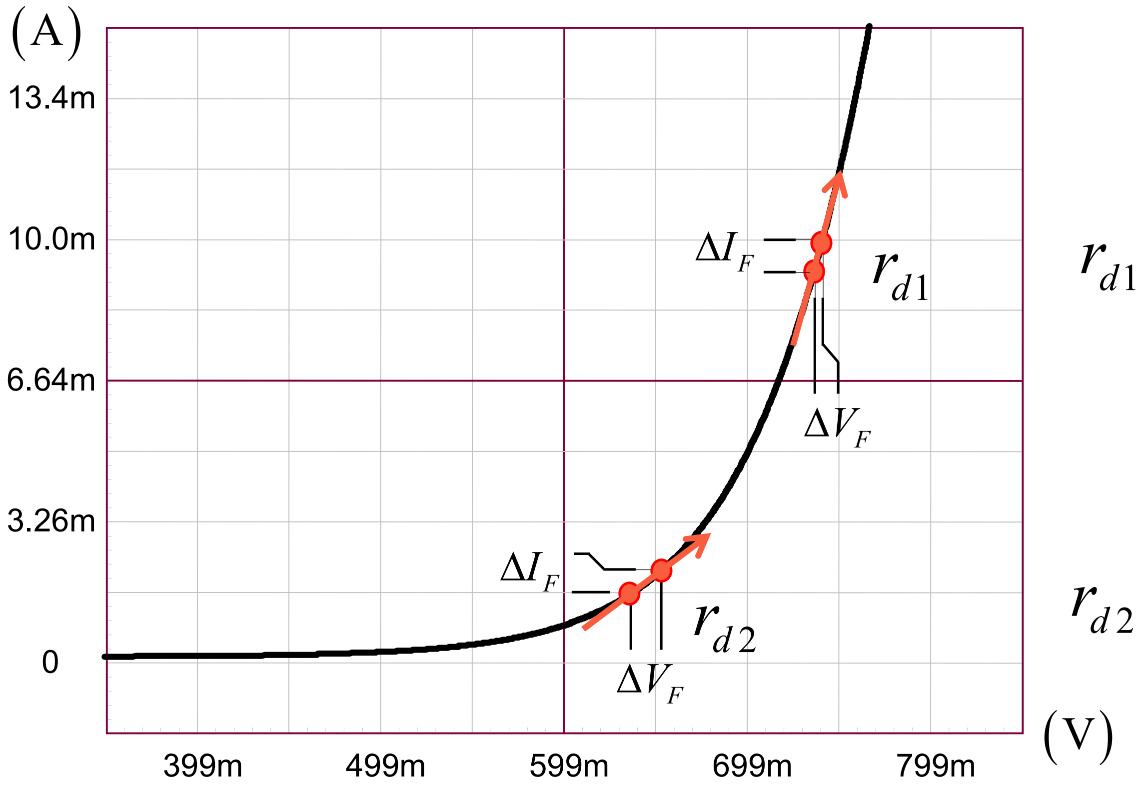
- Isolate the ac term

$$\hat{i}_F = I_F \frac{\hat{v}_F}{nV_T} \Rightarrow r_d = \frac{\hat{v}_F}{\hat{i}_F} = \frac{nV_T}{I_F} [\Omega]$$

- r_d is the small-signal model of the diode at a current I_F

Method 1, Equation is Perturbed

- The dynamic resistance r_d is the tangent slope



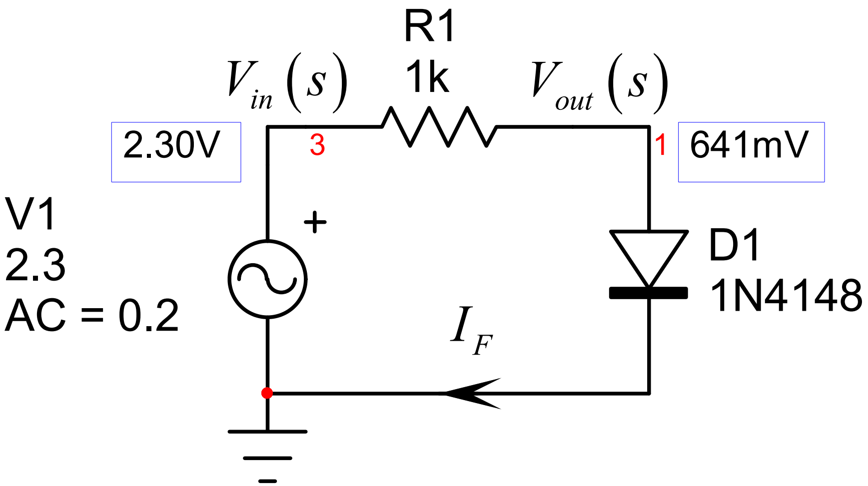
$$r_{d1} = \left.\frac{\Delta V_F}{\Delta I_F}\right|_{I_{F1}}$$

$$r_{d2} = \left.\frac{\Delta V_F}{\Delta I_F}\right|_{I_{F2}}$$

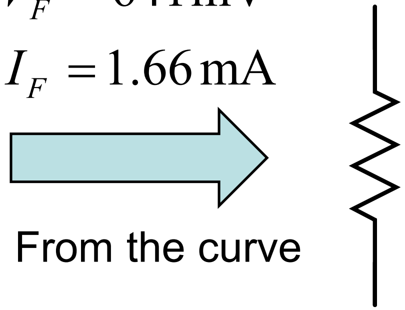
- It is evaluated at a given operating point

Replace the Nonlinear Model

- For the transfer function, compute the small-signal model

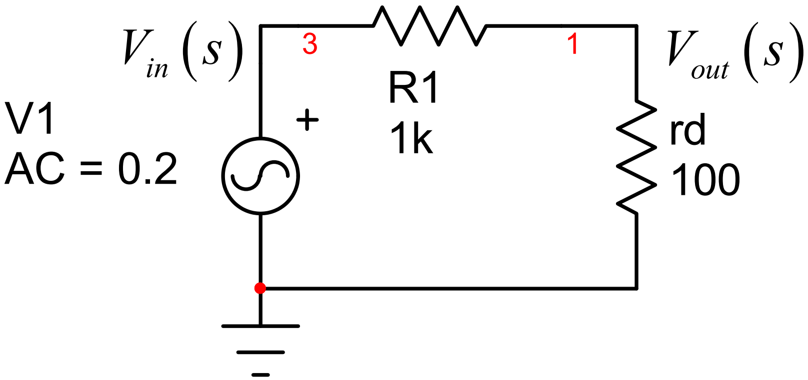


$$V_F = 641\,\mathrm{mV}$$
$$I_F = 1.66\,\mathrm{mA}$$



$$r_d = 100\,\Omega$$

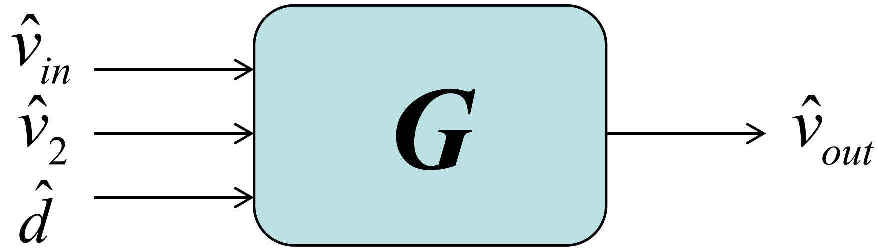
- Plug it into the circuit and solve the equations



$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{r_d}{r_d + R_1} \approx 91m$$

Method 2, Use Partial Differentiation

- Sometimes, sorting out ac and dc equations is tricky
- Partial derivative is an easy way to get ac-only terms
- There is an analogy with the superposition theorem



- You compute the output value while 2 inputs are zero

$$\hat{v}_{out} = G_{V_{out},V_{in}}\Big|_{V_2,d}\hat{v}_{in} + G_{V_{out},V_2}\Big|_{V_{in},d}\hat{v}_2 + G_{V_{out},d}\Big|_{V_{in},V_2}\hat{d}$$

- The total response is the sum of the individual responses

Partial Differentiation

- A function depends on multiple variables $f(x,y,z)$
- We can calculate the rate of change of this function when:
  - x varies while y and z are constant $\frac{df(x)}{dx}$
  - y varies while x and z are constant $\frac{df(y)}{dy}$
  - z varies while x and y are constant $\frac{df(z)}{dz}$
- Mathematically, we perform a partial differentiation, noted:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Total rate of change → df

Partial rates of change

Partial Differentiation

- With one variable, partial and total derivatives are similar

$$f'(x) = \frac{df(x)}{dx} = \frac{\partial f(x)}{\partial x}$$

- Partial derivative is faster with complex equations

$$I_{out} = \frac{L_p \left(\frac{V_c}{R_i}\right)^2 F_{sw}}{2V_{out}}$$

- You have three variables, you can perturb and collect…

$$I_{out} + \hat{i}_{out} = \frac{L_p \left(\frac{V_c + \hat{v}_c}{R_i}\right)^2 \left(F_{sw} + \hat{f}_{sw}\right)}{2\left(V_{out} + \hat{v}_{out}\right)}$$

→

$I_{out} = \ldots$ dc equation

$\hat{i}_{out} = \ldots$ ac equation

Partial Differentiation

- … or you can use Mathcad® and automate the process

$$\hat{i}_{out} = \frac{\partial I_{out}\left(F_{sw},V_c,V_{out}\right)}{\partial F_{sw}}\hat{f}_{sw} + \frac{\partial I_{out}\left(F_{sw},V_c,V_{out}\right)}{\partial V_c}\hat{v}_c + \frac{\partial I_{out}\left(F_{sw},V_c,V_{out}\right)}{\partial V_{out}}\hat{v}_{out}$$

- You have the result in a twinkling of an eye

$$\hat{i}_{out} = \frac{L_p V_c^2}{2R_i^2 V_{out}}\hat{f}_{sw} + \frac{F_{sw} L_p V_c}{R_i^2 V_{out}}\hat{v}_c - \frac{F_{sw} L_p V_c^2}{2R_i^2 V_{out}^2}\hat{v}_{out}$$

- We can apply this technique to our diode equation

$$\hat{i}_F = \frac{\partial\left(I_S e^{\frac{V_F}{nV_T}}\right)}{\partial V_F}\hat{v}_F = I_S e^{\frac{V_F}{nV_T}}\frac{\hat{v}_F}{nV_T} = \frac{I_F}{nV_T}\hat{v}_F \Rightarrow r_d = \frac{nV_T}{I_F}[\Omega]$$

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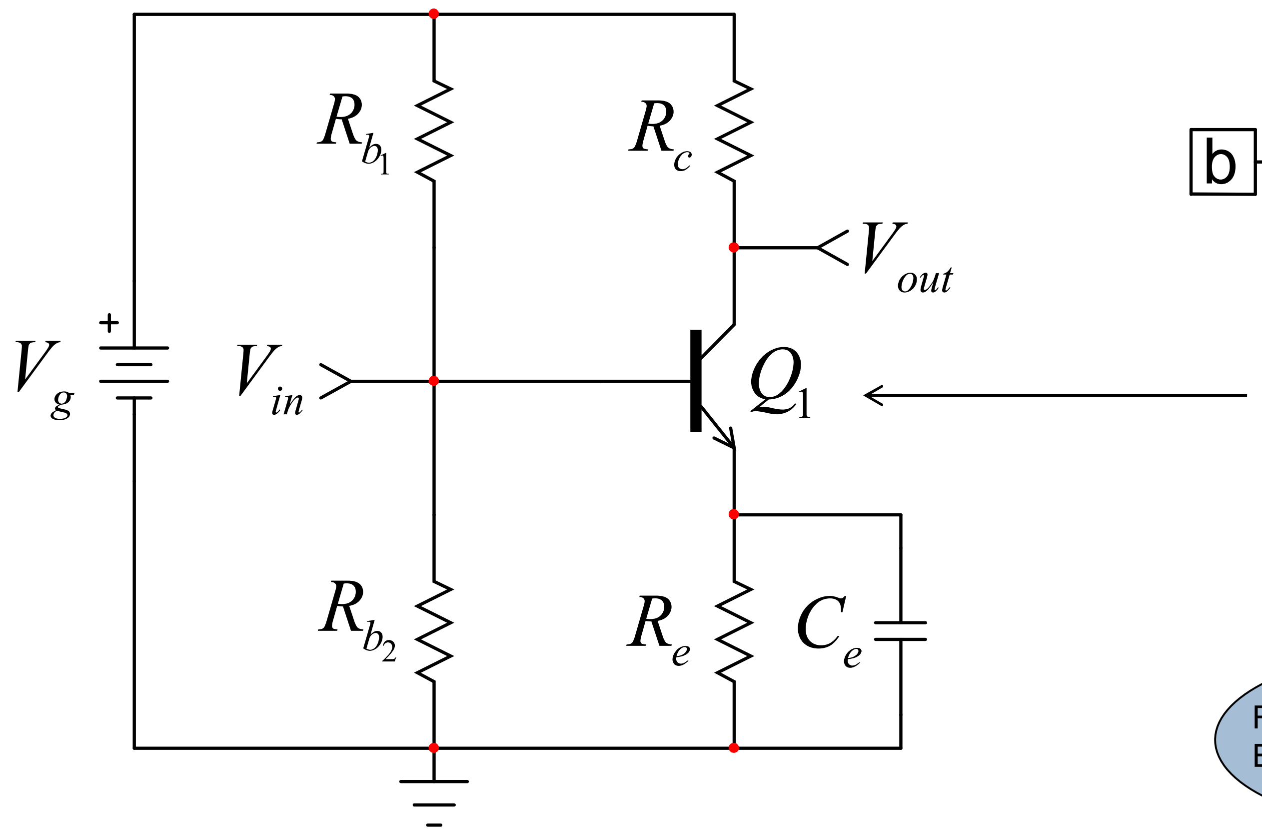
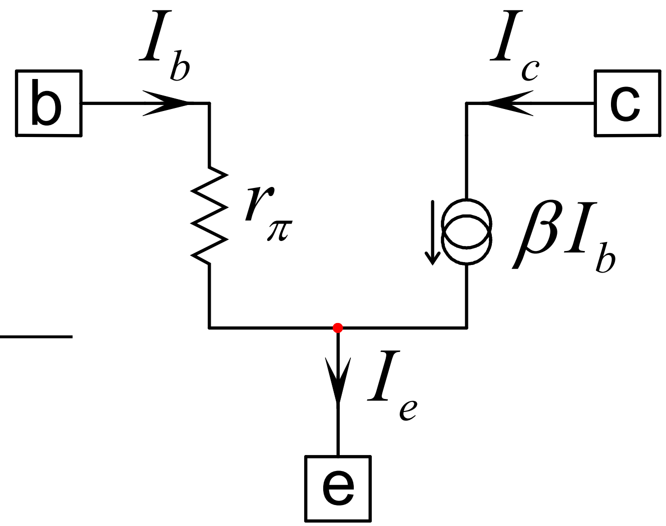
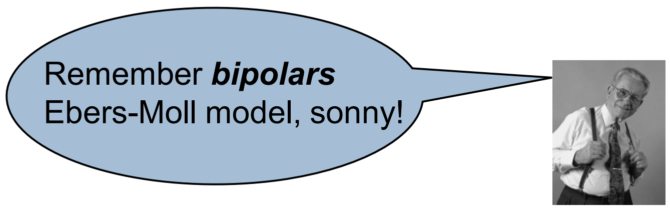
The Bipolar Small-Signal Model

- A bipolar transistor is a highly non-linear system
- Replace it by its small-signal model to get the response

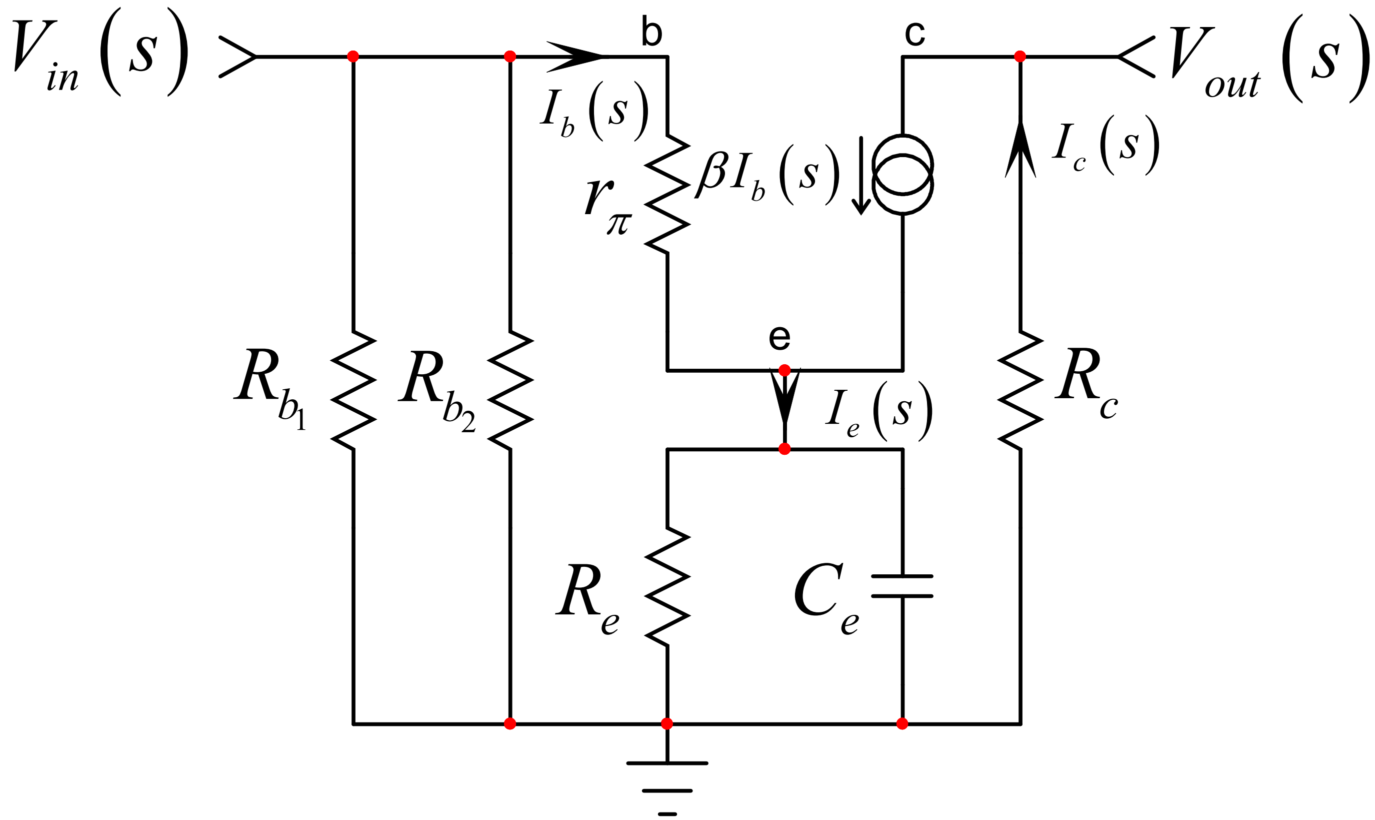






A Linear System

- The new circuit is a small-signal linear architecture



- Laplace notation applies to get the transfer function

The Transfer Function

- A transfer function links a response to an excitation

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} \quad \begin{matrix} \longleftarrow \text{response} \\ \longleftarrow \text{excitation} \end{matrix} \qquad H(s) = \frac{N(s)}{D(s)} \quad \begin{matrix} \longleftarrow \text{zeros} \\ \longleftarrow \text{poles} \end{matrix}$$

- It must be written in a so-called low entropy form

$$H(s) = \frac{b_0 + b_1 s}{a_0 + a_1 s} \quad \xrightarrow[\text{factor } a_0]{\text{factor } b_0} \quad \frac{b_0}{a_0} \frac{1 + s\frac{b_1}{b_0}}{1 + s\frac{a_1}{a_0}} \longrightarrow G_0 \frac{1 + s/\omega_{z_1}}{1 + s/\omega_{p_1}}$$

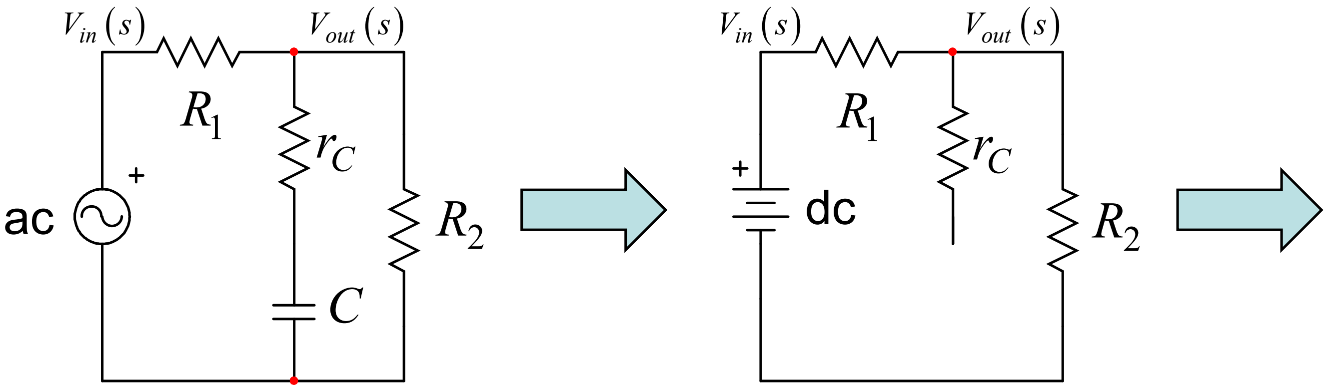
- In this 1st-order expression you can identify the terms

$$G_0 = \frac{b_0}{a_0} \qquad \omega_{z_1} = \frac{b_0}{b_1} \qquad \omega_{p_1} = \frac{a_0}{a_1}$$

dc gain, $s = 0$ — zero — pole

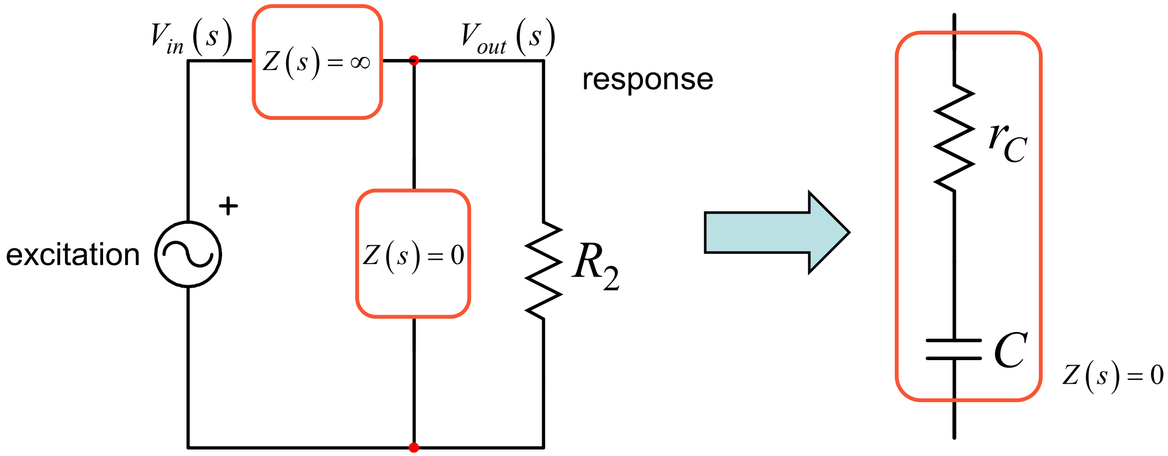
Finding the Zeros

- G_0 is found by shorting inductors and opening capacitors



$$G_0 = \frac{R_2}{R_2 + R_1}$$

- Zeros prevent the excitation from reaching the output



$$r_C + \frac{1}{sC} = \frac{sr_C C + 1}{sC}$$

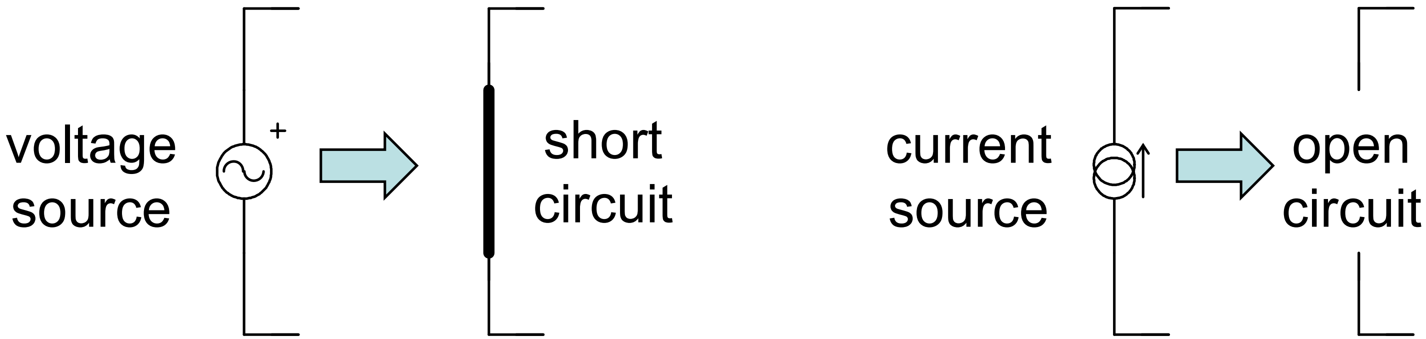
$$sr_C C + 1 = 0$$

$$\omega_{z_1} = \frac{1}{r_C C}$$

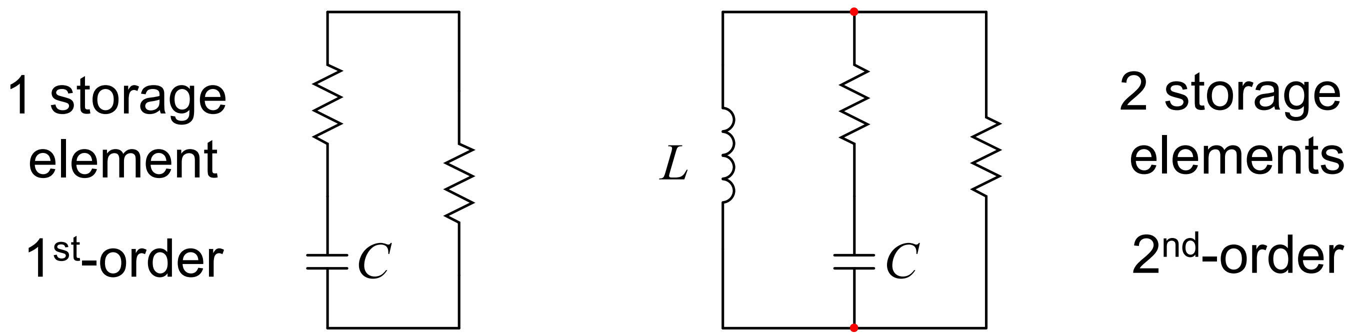
Finding the Poles

- ❑ The poles are linked to the time constants of the system
- ❑ These time constants solely depend on the structure

→ Remove the excitation signal to isolate the structure

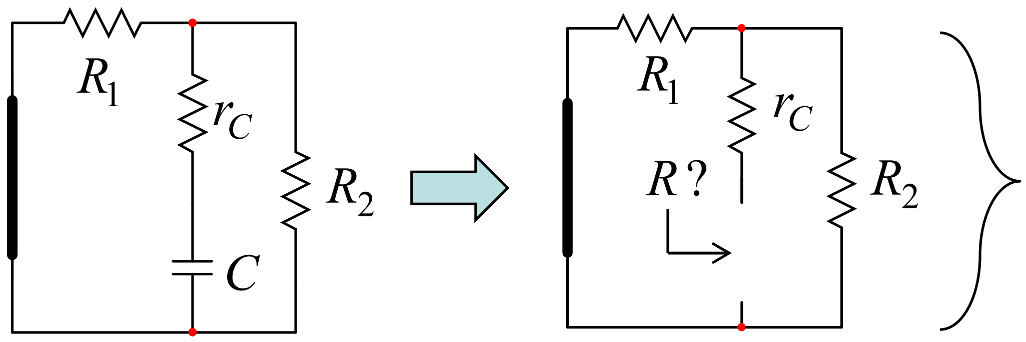


- ❑ The denominator order depends on the storage elements



The Final Answer

- Short the voltage source and calculate the time constant



$$R = r_C + R_1 \| R_2$$

$$\tau = \left(r_C + R_1 \| R_2\right)C$$

$$\omega_{p_1} = \frac{1}{\left(r_C + R_1 \| R_2\right)C}$$

- The complete transfer function is obtained in 3 steps

$$H(s) = G_0 \frac{1 + s/\omega_{z_1}}{1 + s/\omega_{p_1}} = \frac{R_2}{R_2 + R_1} \frac{1 + s r_C C}{1 + s\left(r_C + R_1 \| R_2\right)C}$$

- We derived the transfer function by inspection!

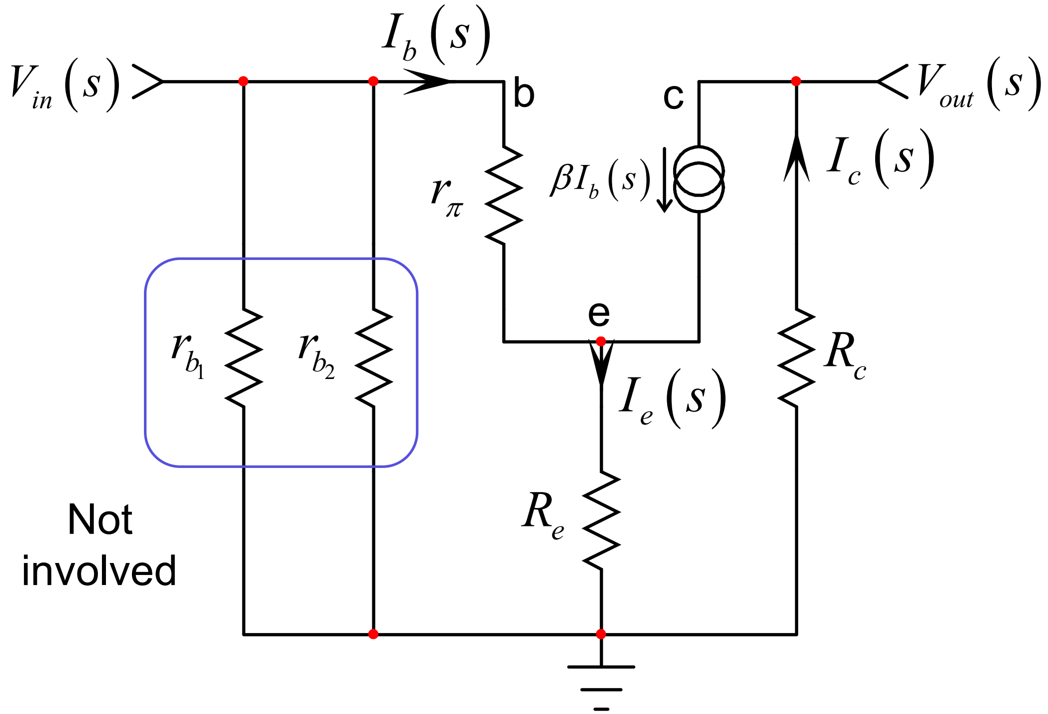
"Fast Analytical Techniques for Electrical and Electronic Circuits", V. Vorpérian, Cambridge Press, 2002

The Transistor Amplifier

- We have just one capacitor, this is a 1st-order system

$$\Rightarrow \quad \frac{V_{out}(s)}{V_{in}(s)} = G_0 \frac{1+s/\omega_{z_1}}{1+s/\omega_{p_1}}$$

- G_0 is found by opening the capacitor



$$V_{out}(s) = -R_c I_c(s) = -R_c \beta I_b(s)$$

$$I_b(s) = \frac{V_{be}(s)}{r_\pi} = \frac{V_{in}(s) - I_e(s) R_e}{r_\pi}$$

$$I_e(s) = (\beta+1) I_b(s)$$

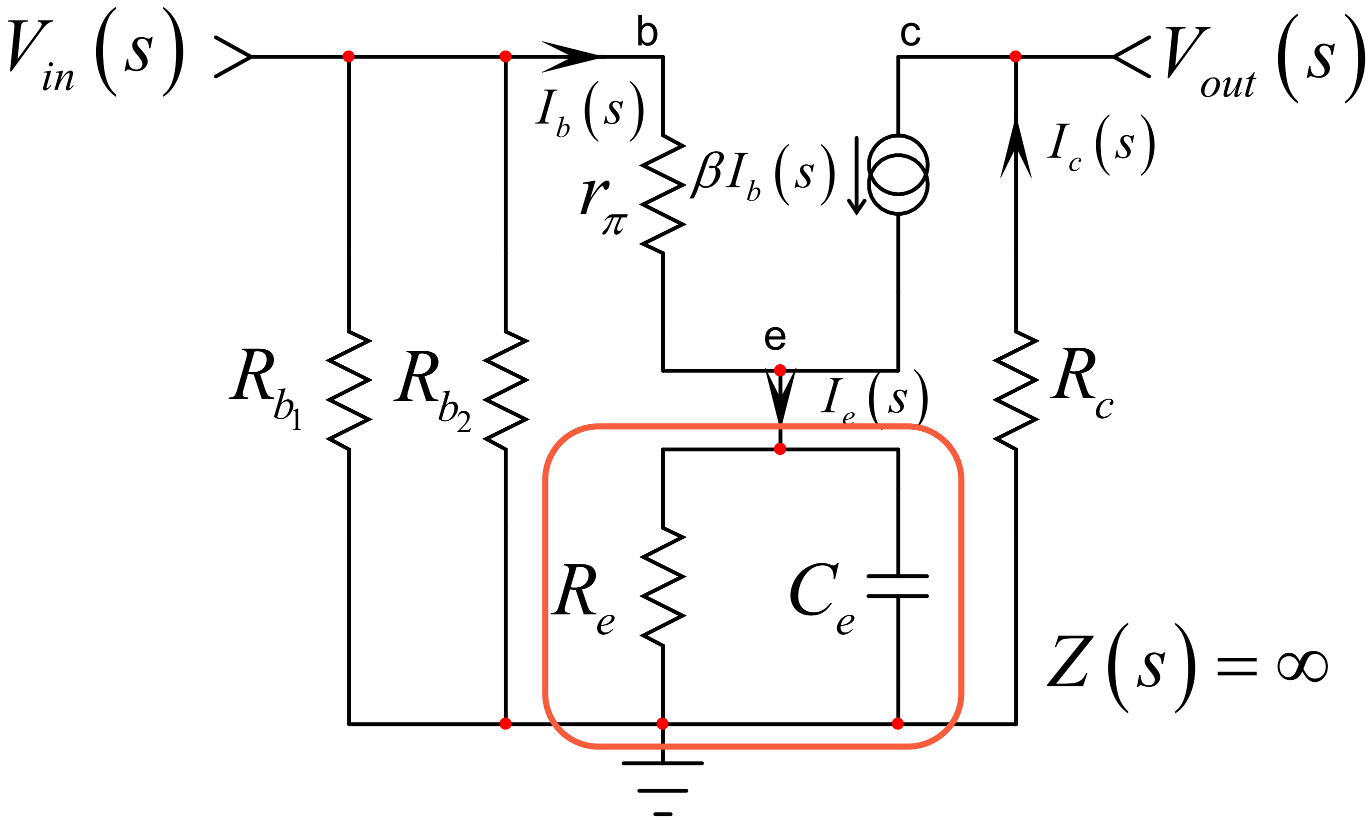
$$I_b(s) = \frac{V_{in}(s) - (\beta+1) I_b(s) R_e}{r_\pi}$$

$$I_b(s) = \frac{V_{in}(s)}{r_\pi + (\beta+1) R_e}$$

$$G_0 = -\frac{R_c \beta}{r_\pi + (\beta+1) R_e} \approx -\frac{R_c}{R_e}$$

Looking for the Zeros

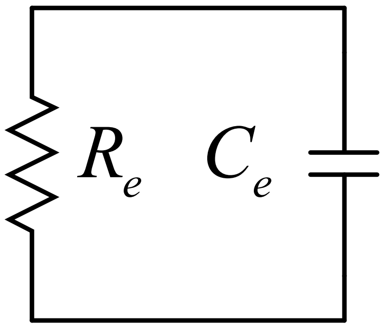
- ❑ What could prevent the excitation from reaching the output?



- ❑ When this impedance is infinite, there is no base current

Looking for the Zeros

- The frequency where the impedance is infinite is our zero



$$Z(s) = R_e \,||\, \frac{1}{sC_e} = \frac{R_e}{1 + sR_eC_e}$$

- When the denominator is equal to zero, Z is infinite

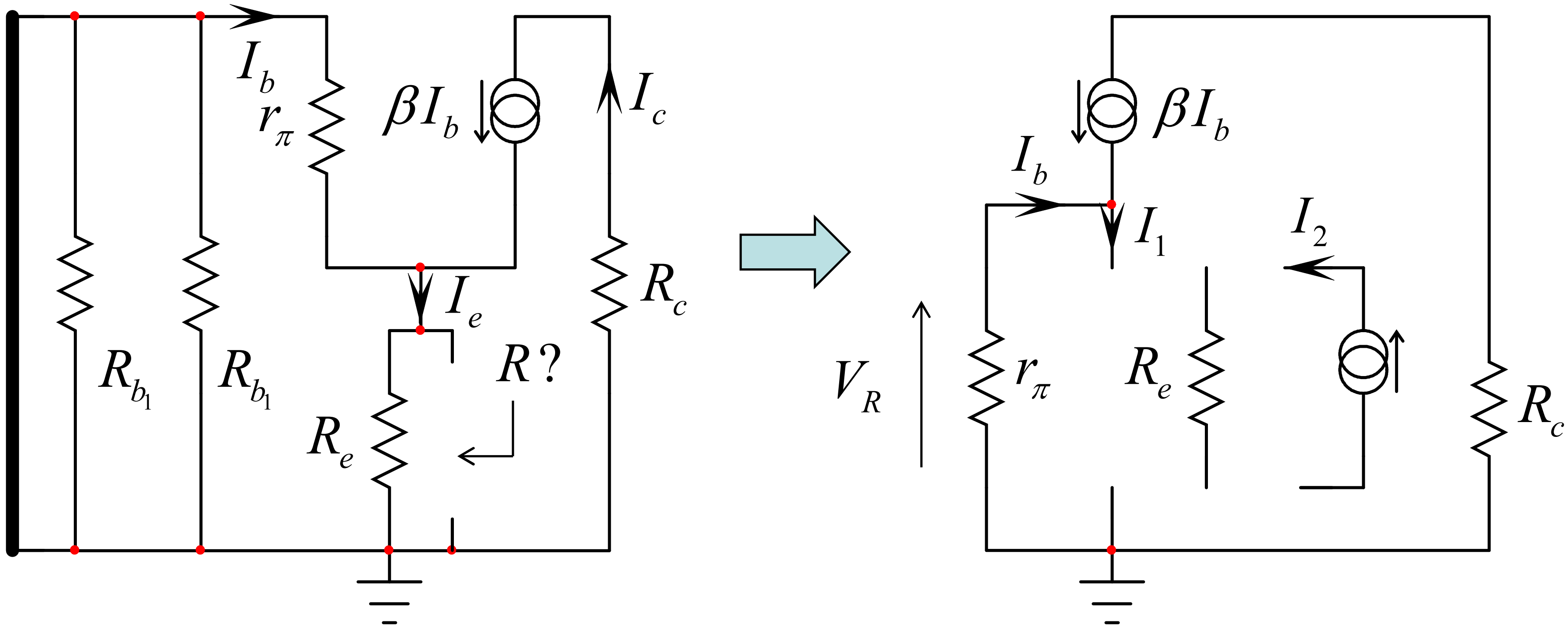
$$1 + sR_eC_e = 0 \quad \Rightarrow \quad \omega_{z_1} = \frac{1}{R_eC_e}$$

- We are almost there

$$\frac{V_{out}(s)}{V_{in}(s)} = -\frac{R_c}{R_e}\frac{1 + sR_eC_e}{D(s)}$$

Looking for the Poles

- For the system time constant, short the excitation



- Write two simple equations

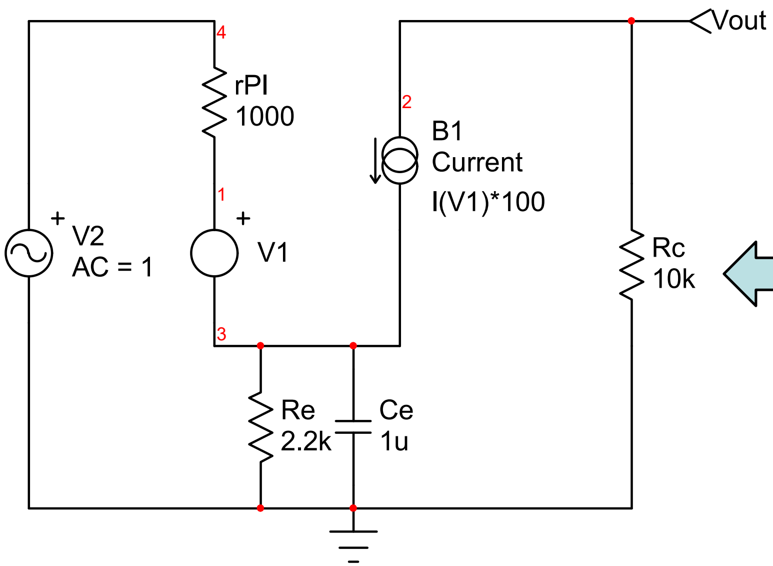
$$I_b = -\frac{V_R}{r_\pi} \qquad I_1 = (\beta+1) I_b = -I_2 \qquad \Rightarrow \qquad R = \frac{V_R}{I_2} = R_e \,||\, \left(\frac{r_\pi}{\beta+1}\right)$$

The Final Transfer Function

- We have our transfer function in a few steps only

$$\frac{V_{out}(s)}{V_{in}(s)} \approx -\frac{R_c}{R_e}\frac{1+sR_eC_e}{1+sC_eR_e \parallel \left(\frac{r_\pi}{\beta+1}\right)}$$

- The next step is to check maths versus simulation



$R_e := 2.2k\Omega \quad r_{PI} := 1k\Omega \quad R_c := 10k\Omega \quad C_e := 1\mu F \quad \beta := 100 \quad \|(x,y) := \frac{x \cdot y}{x+y}$

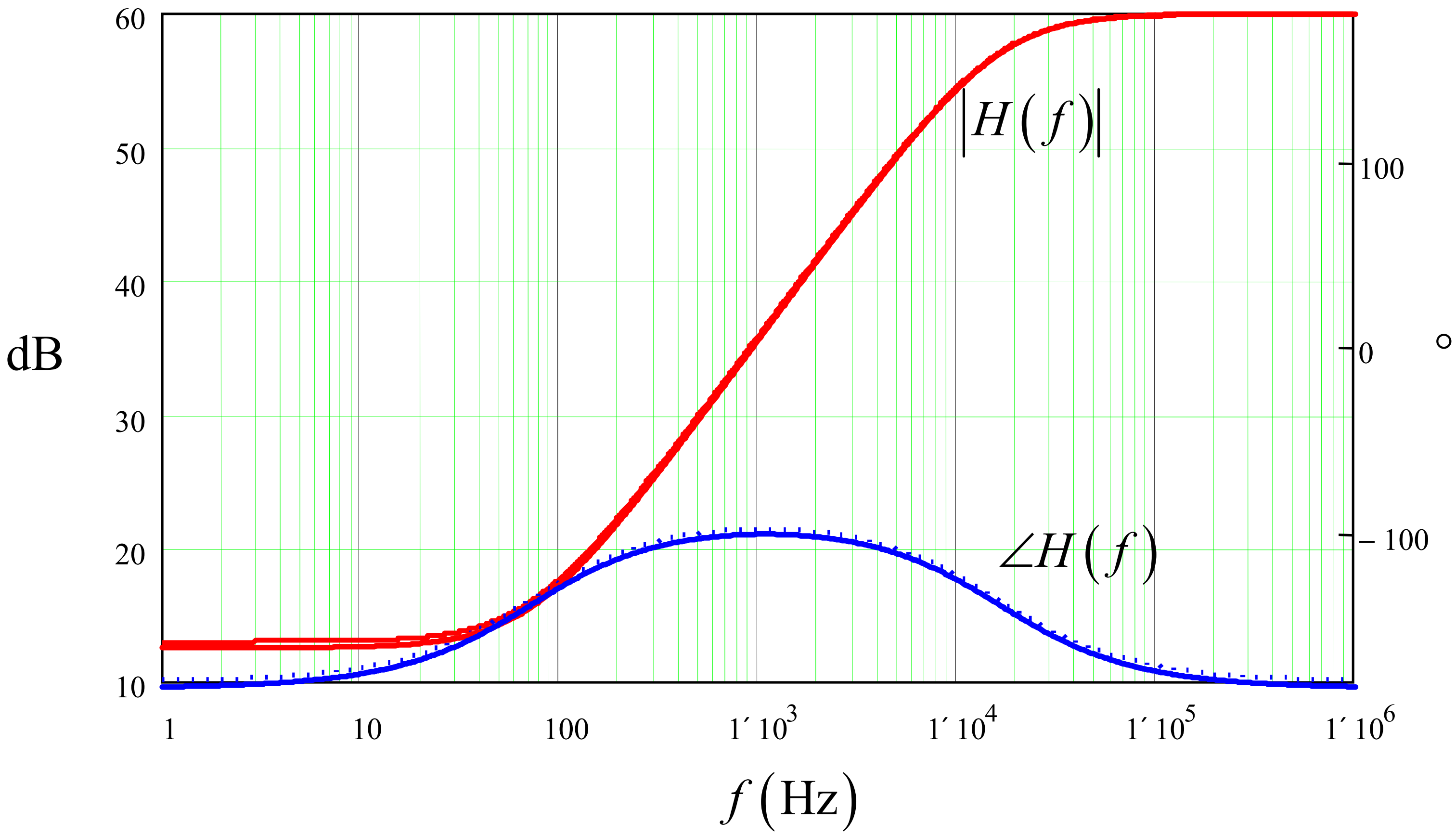
$\omega_{z1} := \frac{1}{R_e \cdot C_e} \quad f_{z1} := \frac{\omega_{z1}}{2 \cdot \pi} = 72.343\,Hz \quad G_0 := \frac{R_c \cdot \beta}{r_{PI} + (\beta+1) \cdot R_e} = 4.48$

$\omega_{p1} := \frac{1}{\left[R_e \parallel \left(\frac{r_{PI}}{\beta+1}\right)\right] \cdot C_e} \quad f_{p1} := \frac{\omega_{p1}}{2\pi} = 16.147\,kHz \quad H_1(s) := -G_0 \cdot \frac{1+\frac{s}{\omega_{z1}}}{1+\frac{s}{\omega_{p1}}}$

Mathcad®

The Final Transfer Function

- ❑ The curves superimpose, calculations are good!



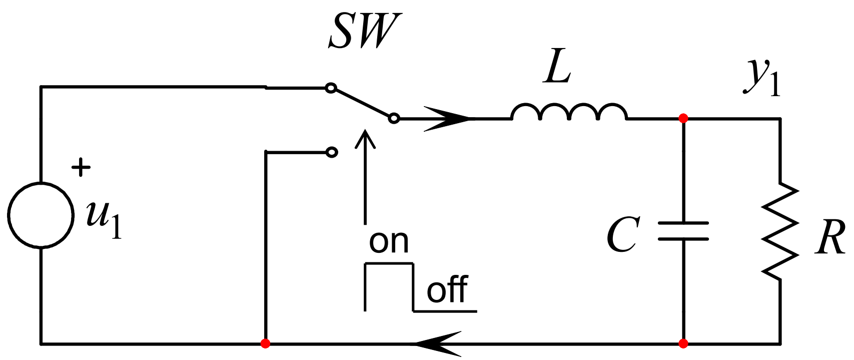
- ❑ Always run this sanity check to verify your results

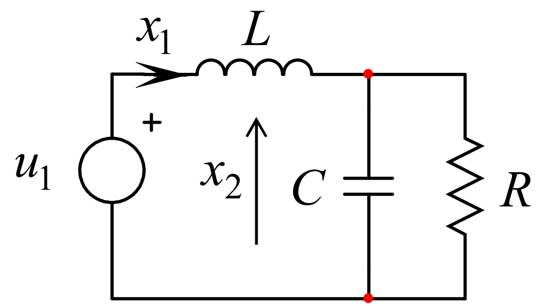
Agenda

- ❑ Linear and Non-Linear Functions
- ❑ What is a Small-Signal Model?
- ❑ Fast Analytical Techniques at Work
- ❑ **From a Switched to Linearized Model**
- ❑ The CCM VM Small-Signal PWM Switch Model
- ❑ The DCM VM Small-Signal PWM Switch Model
- ❑ Peak Current Mode Control in Large Signal
- ❑ The CCM CM Small-Signal PWM Switch Model
- ❑ The DCM CM Small-Signal PWM Switch Model
- ❑ The PWM Switch in Boundary Mode

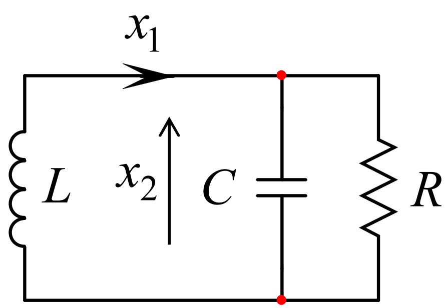
Non-Linearity in a Switching Converter

- A switching converter is ruled by linear equations…





during
DT_{sw}

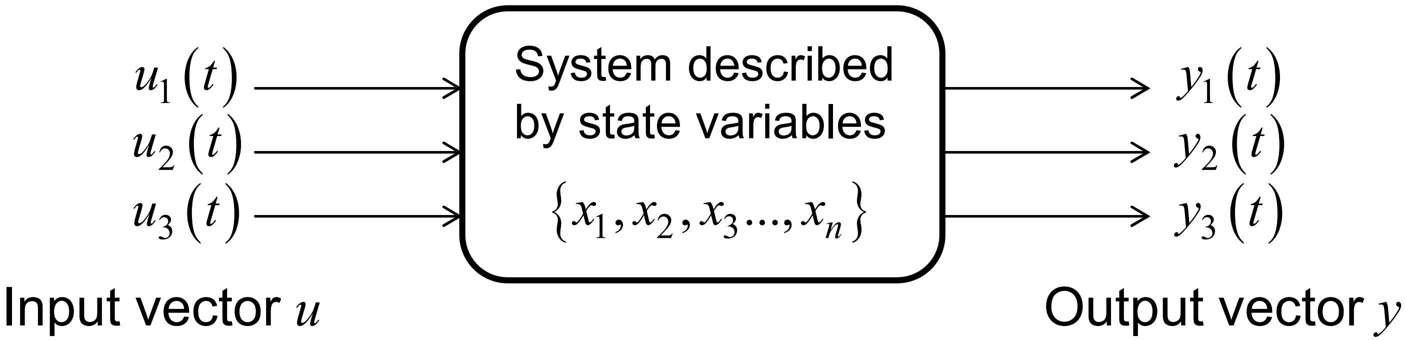


during
$(1-D)T_{sw}$

- …combining so-called state variables

State Variables

- State variables describe the mathematical state of a system
- n state variables for n independent storage elements
- knowing variables state at t_0 helps compute outputs for $t > t_0$



- x_1 is the inductor current and x_2 is the capacitor voltage
- Differentiation gives the state variable rate of change

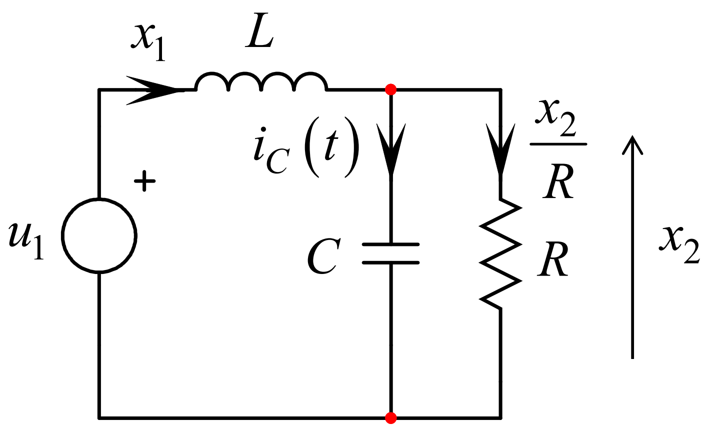
$$x_1 = i_L(t) \qquad \dot{x}_1 = \frac{di_L(t)}{dt}$$

Predict future system state

$$x_2 = v_C(t) \qquad \dot{x}_2 = \frac{dv_C(t)}{dt}$$

Describe the System During the On-Time

- Observe the system during the on-time duration or dT_{SW}:



$$i_C(t) = C\frac{dv_C(t)}{dt} = C\dot{x}_2 = x_1 - \frac{1}{R}x_2$$

$$u_1 = L\frac{di_L(t)}{dt} + v_C(t) = L\dot{x}_1 + x_2$$

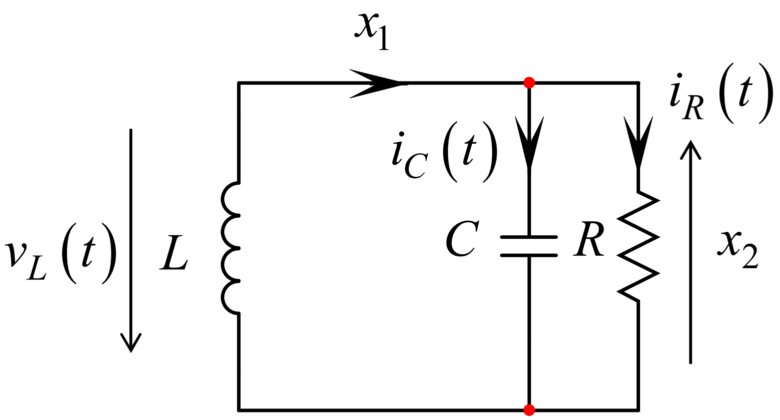
$$\left.\begin{aligned}\dot{x}_1 &= -\frac{1}{L}x_2 + \frac{1}{L}u_1 \\ \dot{x}_2 &= \frac{1}{C}x_1 - \frac{1}{RC}x_2\end{aligned}\right\} \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -1/L \\ 1/C & -1/RC\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1/L & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}u_1 \\ u_2\end{bmatrix}$$

State coefficients

Source coefficients

Describe the System During the Off-Time

- Repeat the exercise during the off-time duration or $1\text{-}dT_{SW}$



$$v_L(t) = -x_2 \quad v_L(t) = L\dot{x}_1$$

$$\Rightarrow \quad x_2 = -L\dot{x}_1$$

$$i_R(t) = x_1 - i_C(t) = x_1 - C\dot{x}_2$$

$$x_2 = i_R(t) R$$

$$\Rightarrow \quad x_2 = R(x_1 - C\dot{x}_2)$$

$$\Rightarrow \quad \left.\begin{aligned} \dot{x}_1 &= 0 - \frac{1}{L}x_2 \\ \dot{x}_2 &= \frac{1}{C}x_1 - \frac{1}{RC}x_2 \end{aligned}\right\} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

State coefficients

Source coefficients

Make it Fit the State Equation Format

- Arrange expressions to make them fit the format:

$$\dot{x} = \mathbf{A}x(t) + \mathbf{B}u(t)$$ State equation

on-time network

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/L & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \mathbf{A}_1 = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \mathbf{B}_1 = \begin{bmatrix} 1/L & 0 \\ 0 & 0 \end{bmatrix}$$

off-time network

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \mathbf{A}_2 = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \mathbf{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- How do we link matrixes $\mathbf{A}_1$ and $\mathbf{A}_2$, $\mathbf{B}_1$ and $\mathbf{B}_2$?
- We smooth the discontinuity by weighting them by D and $1\text{-}D$

$$\dot{x} = \left[\mathbf{A}_1 D + \mathbf{A}_2 (1-D)\right] x(t) + \left[\mathbf{B}_1 D + \mathbf{B}_2 (1-D)\right] u(t)$$

The State-Space Averaging Method (SSA)

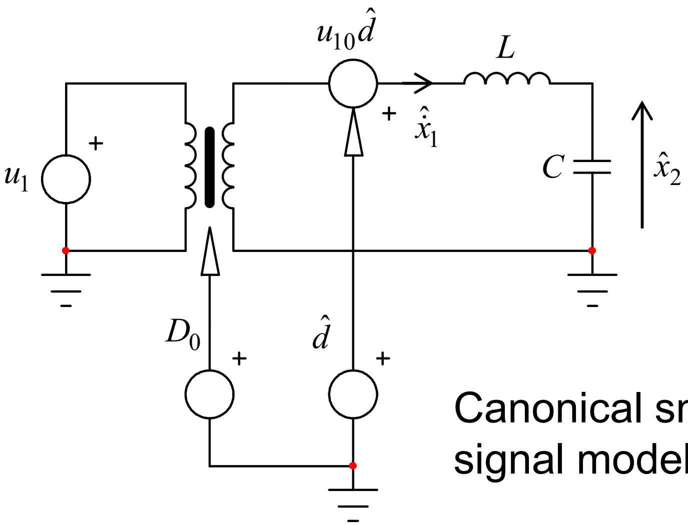
- We now have a continuous large-signal equation
- We need to linearize it via perturbations

$$D = D_0 + \hat{d} \qquad x = x_0 + \hat{x} \qquad u = u_0 + \hat{u}$$

$$\dot{x} = \left[\mathbf{A}_1 D + \mathbf{A}_2 (1 - D)\right] x(t) + \left[\mathbf{B}_1 d + \mathbf{B}_2 (1 - D)\right] u(t)$$

$$\dot{\hat{x}}_1 = \frac{1}{L}\hat{x}_2 + \frac{D_0}{L}\hat{u}_1 + \frac{\hat{d}}{L}u_{10}$$

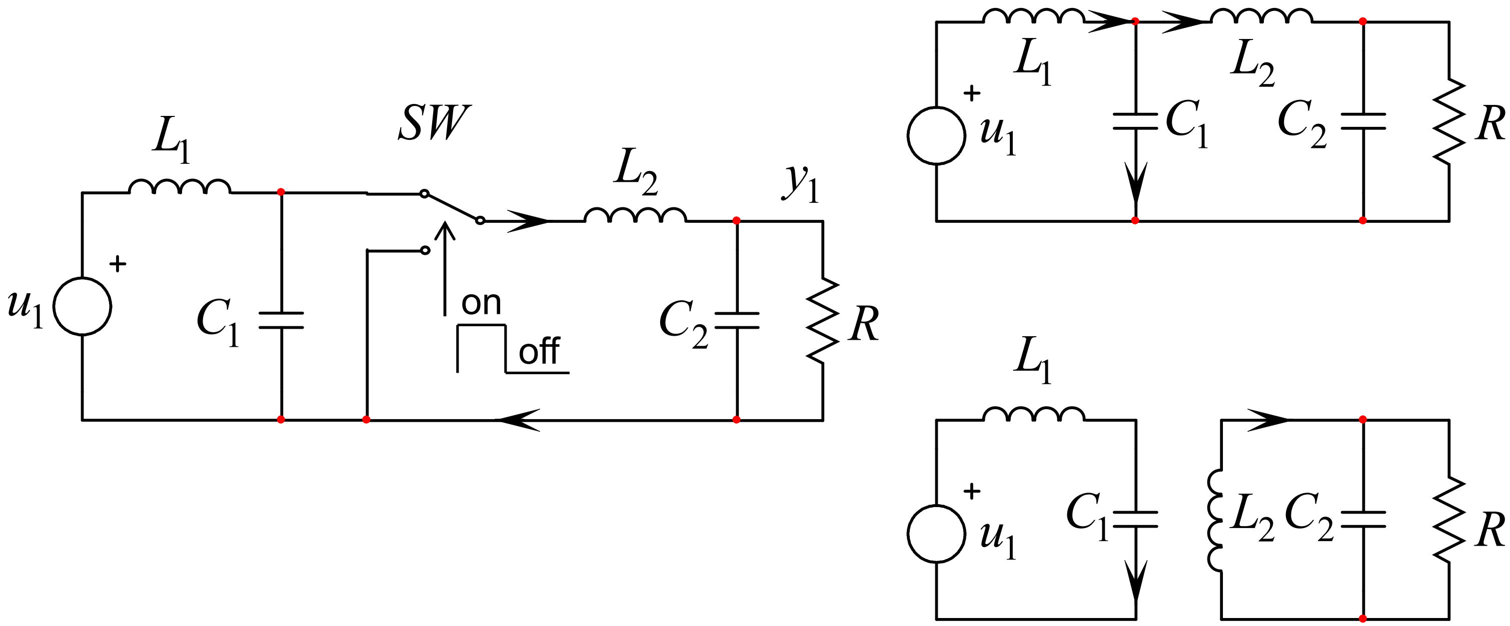
$$\dot{\hat{x}}_2 = \frac{1}{C}\hat{x}_1 - \frac{1}{RC}\hat{x}_2$$



Canonical small-signal model

The State-Space Averaging Method (SSA)

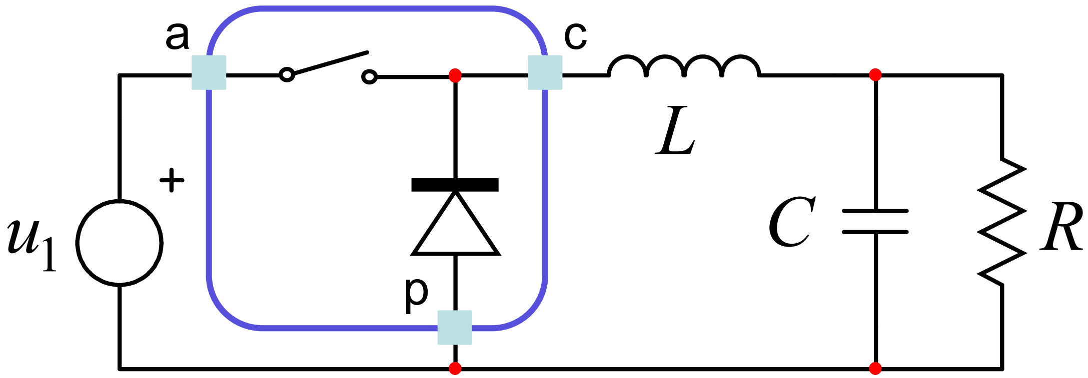
- ❑ SSA was applied to switching converters by Dr Ćuk in 1976
- ❑ It is a long, painful process, manipulating numerous terms
- ❑ What if you add an EMI filter for instance?



- ❑ 4 state variables and you have to re-derive all equations!

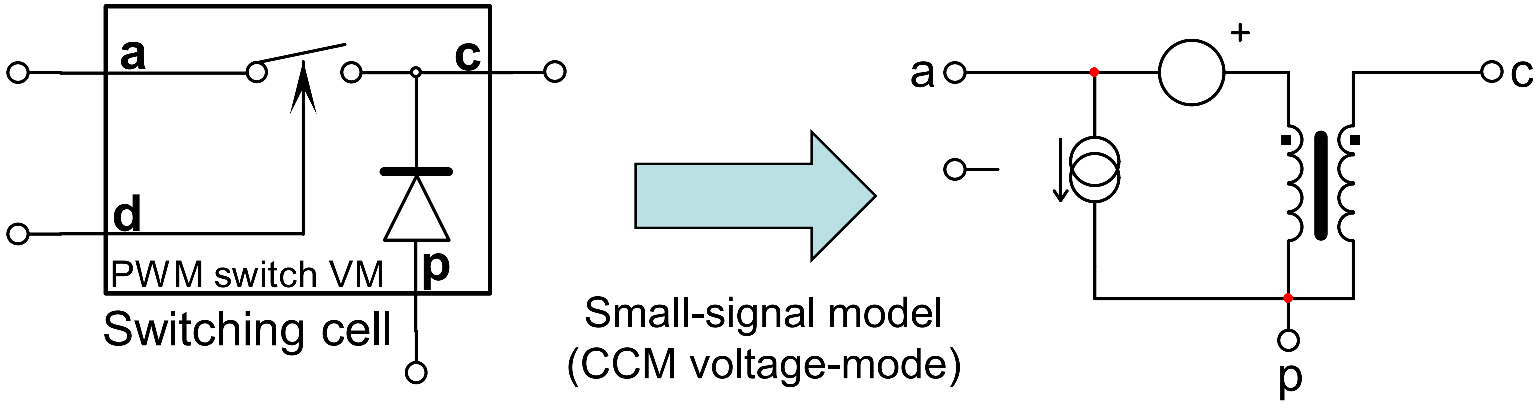
The PWM Switch Model in Voltage Mode

- We know that non-linearity is brought by the switching cell



a: active
c: common
p: passive

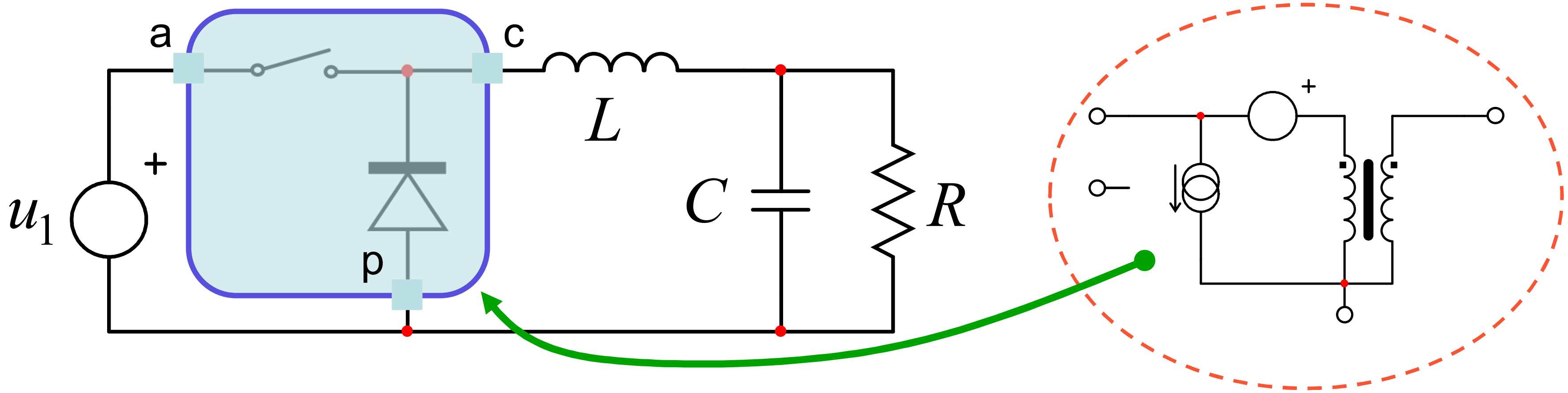
- Why don't we linearize the cell alone?



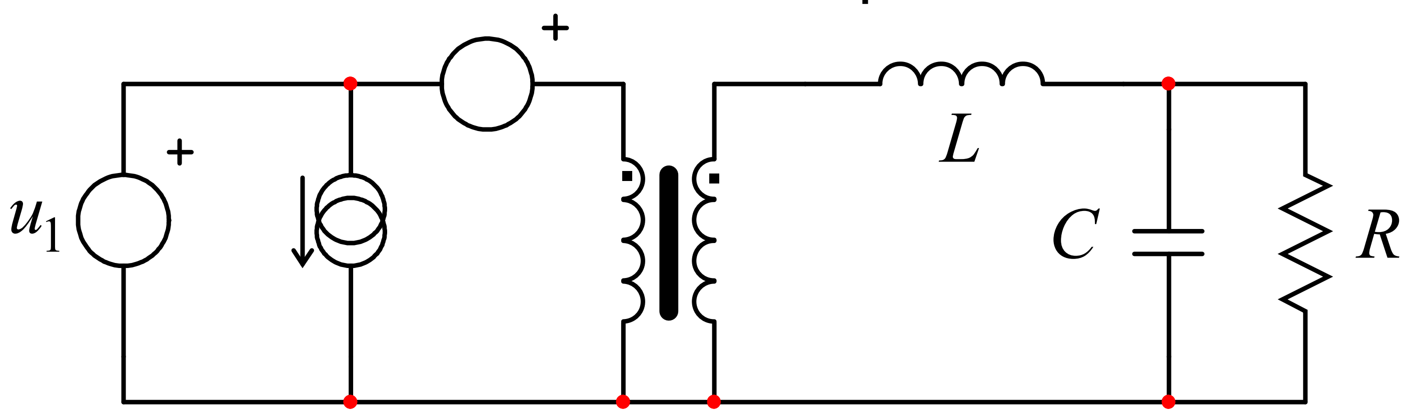
V. Vorpérian, "Simplified Analysis of PWM Converters using Model of PWM Switch, parts I and II" IEEE Transactions on Aerospace and Electronic Systems, Vol. 26, NO. 3, 1990

Replace the Switches by the Model

- Like in the bipolar circuit, replace the switching cell…

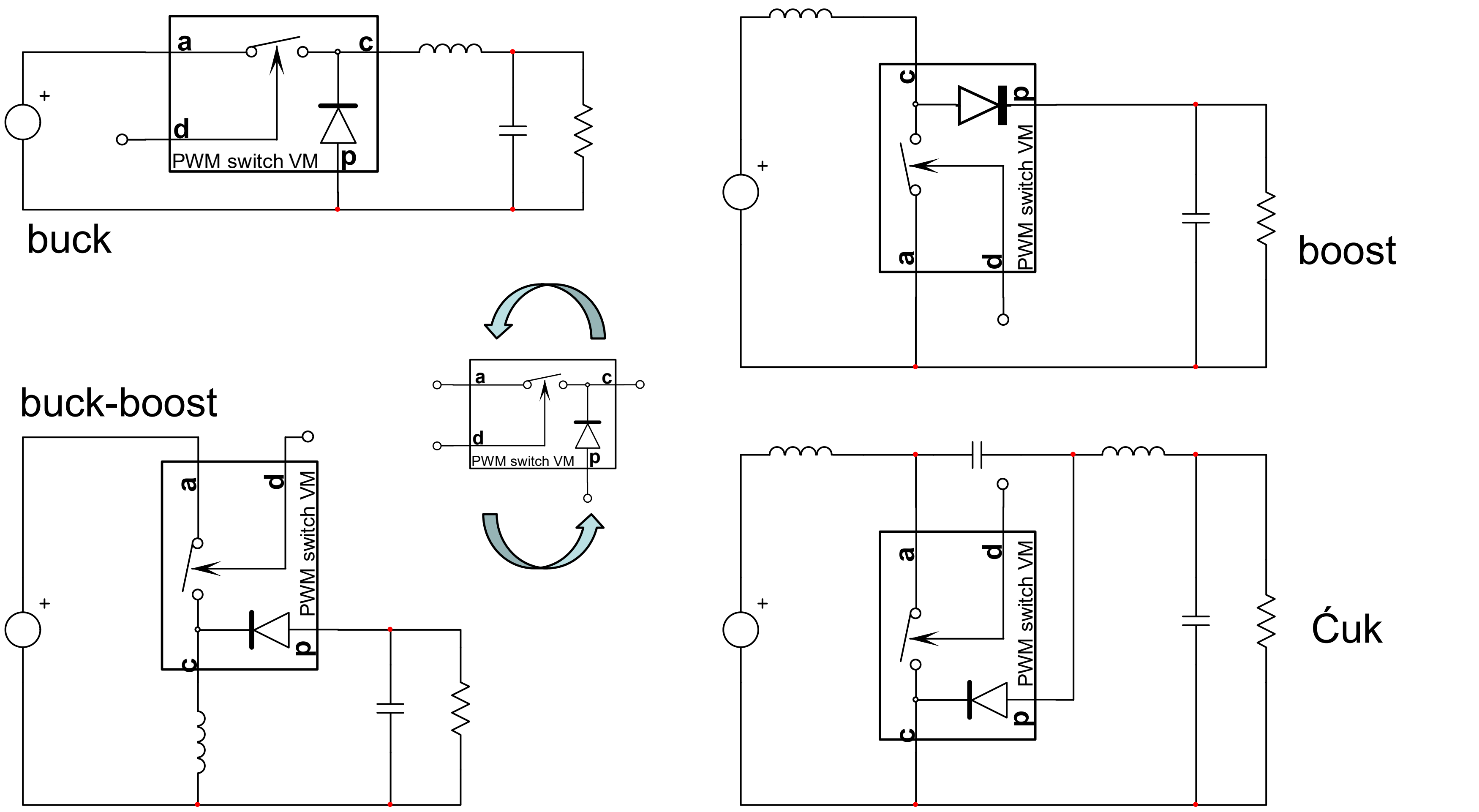


- …and solve a set of linear equations!



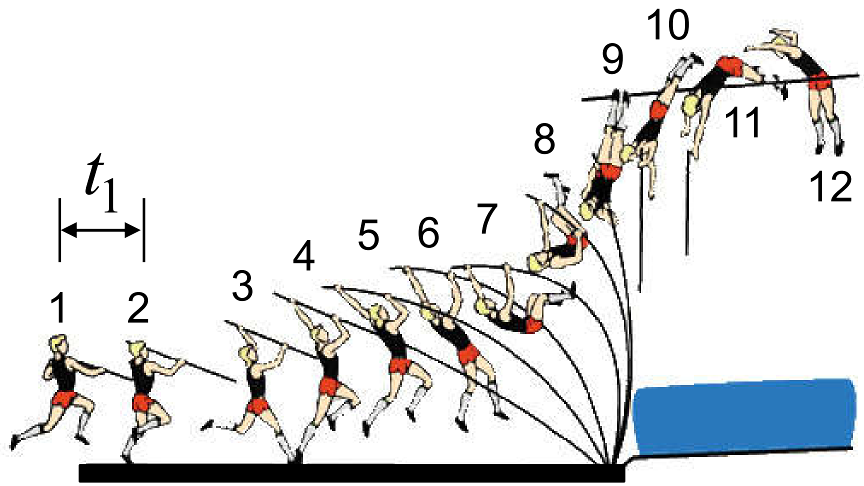
An Invariant Model

- ❑ The switching cell made of two switches is everywhere!



Smoothing the Discontinuity

- A CRT TV displays frames at a certain rate, 50 per second



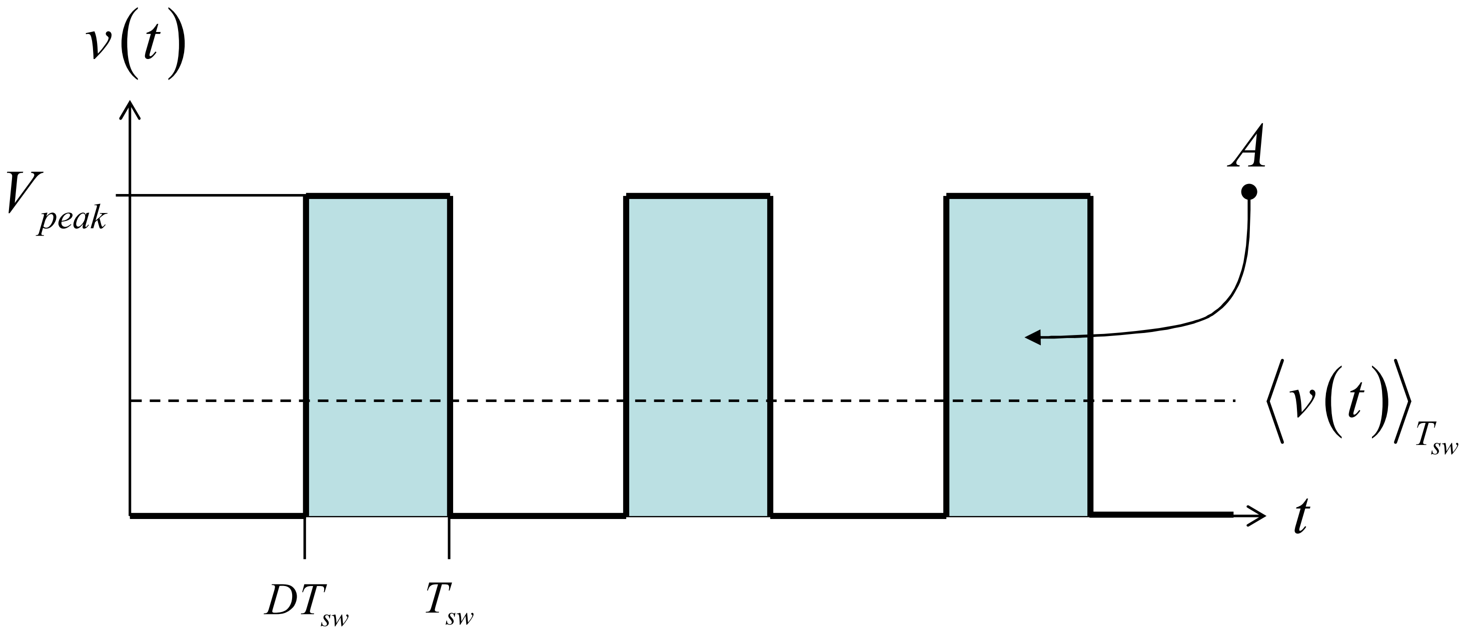
- The optic nerve time constant is larger than an interval
- A succession of discrete events is seen as continuous

Low-frequency filtering
Integration

See "phi phenomena", www.wikipedia.org

Averaging Waveforms

- The keyword in the PWM switch is *averaging*

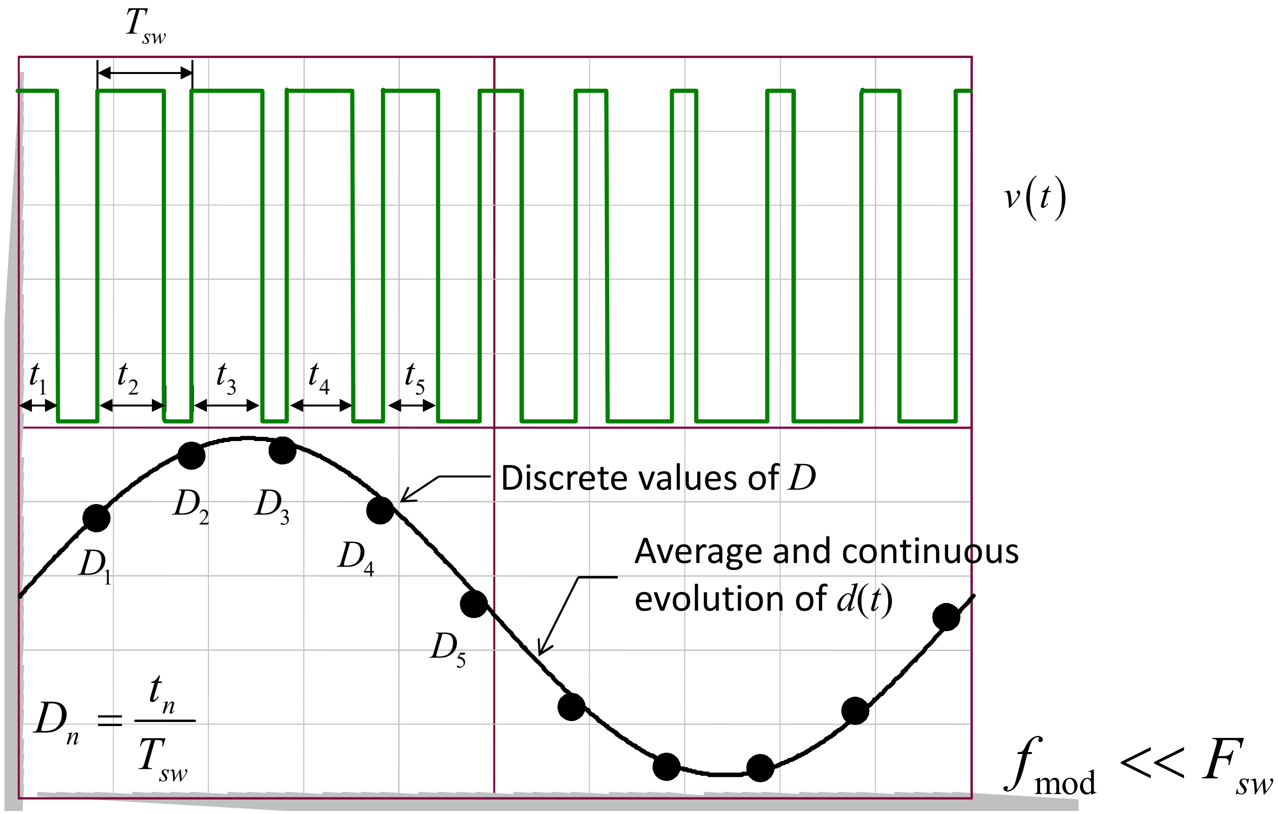


$$\langle v(t)\rangle_{T_{sw}} = \frac{1}{T_{sw}}\int_0^{T_{sw}} v(t)\cdot dt = \frac{1}{T_{sw}}\int_{DT_{sw}}^{T_{sw}} V_{peak}\cdot dt = V_{peak}(1-D) = V_{peak}D'$$

- The resulting function is continuous in time

From Steps to a Continuous Function

❑ Some functions require mathematical abstraction: duty ratio

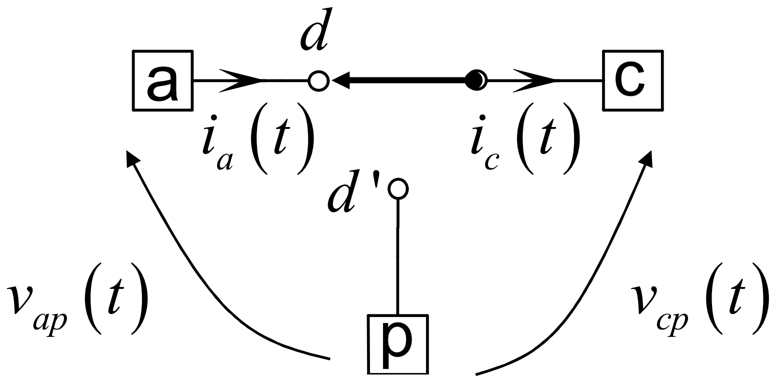


❑ At the modulation frequency scale, points look contiguous

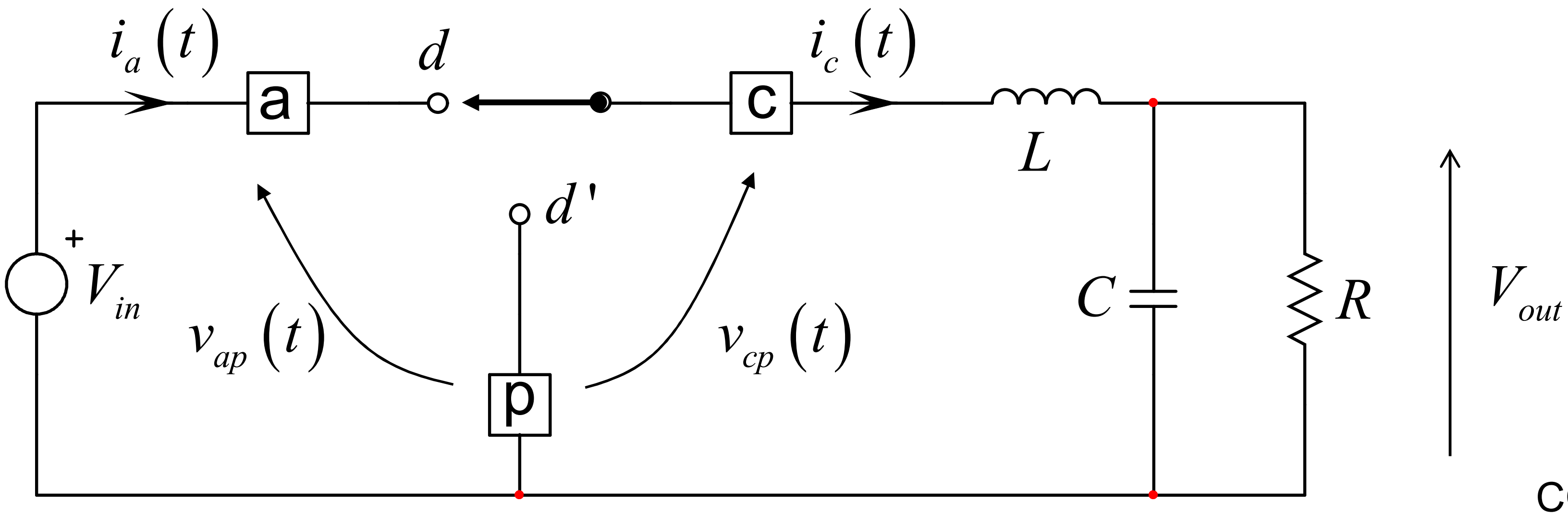
❑ Link them through a continuous-time ripple-free function $d(t)$

The Common Passive Configuration

- The PWM switch is a single-pole double-throw model



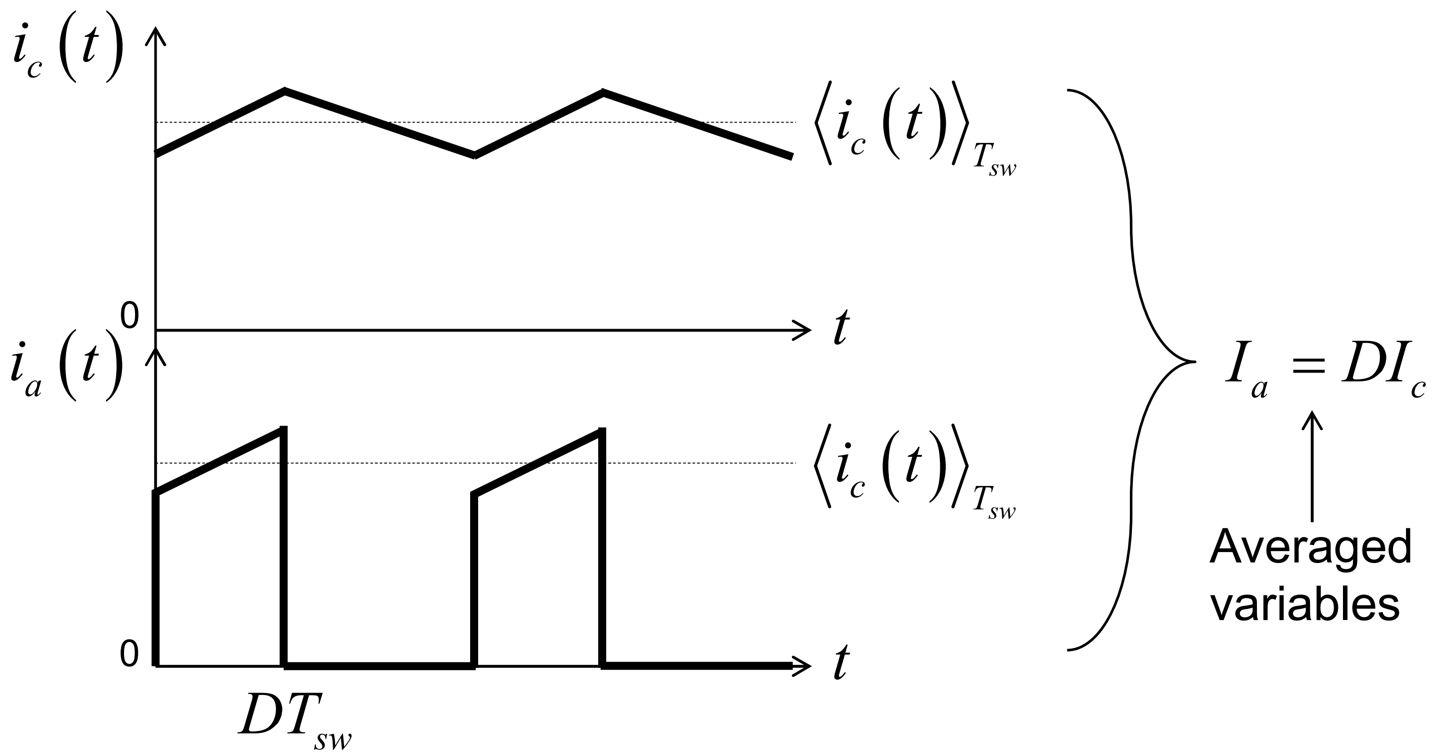
- Install it in a buck converter and draw the waveforms



CCM

The Common Passive Configuration

- ❑ Average the current waveforms across the PWM switch

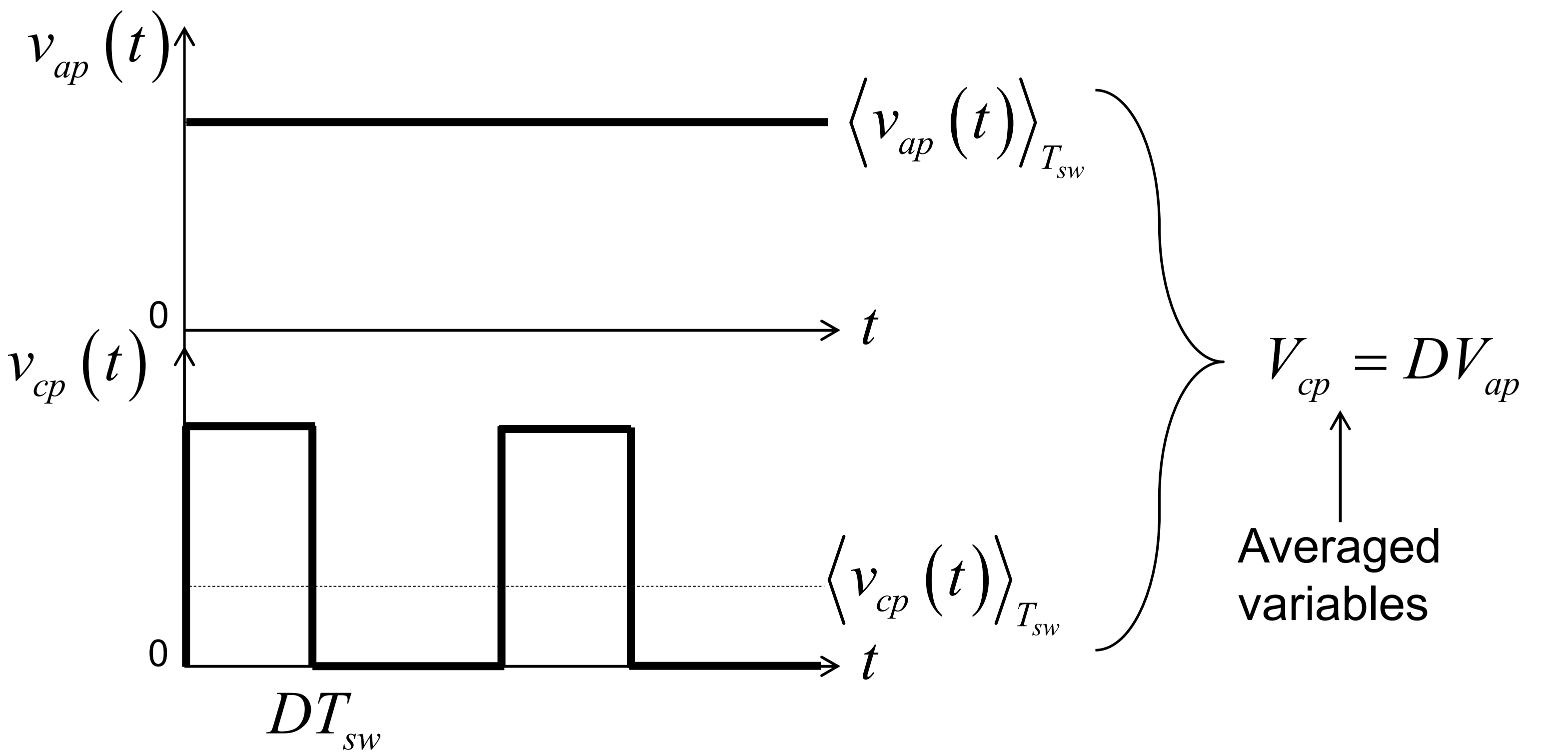


$$\langle i_a(t)\rangle_{T_{sw}} = I_a = \frac{1}{T_{sw}}\int_0^{dT_{sw}} i_a(t)\,dt = D\langle i_c(t)\rangle_{T_{sw}} = DI_c$$

CCM

The Common Passive Configuration

- Average the voltage waveforms across the PWM switch

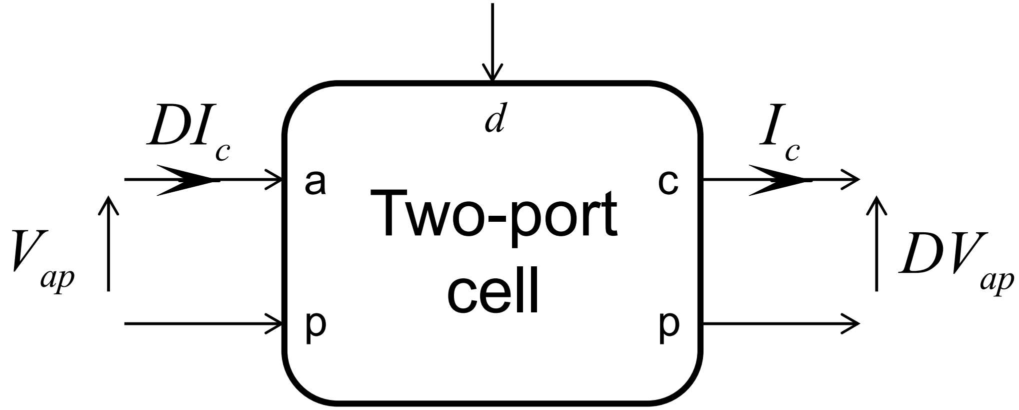


$$\langle v_{cp}(t) \rangle_{T_{sw}} = V_{cp} = \frac{1}{T_{sw}} \int_0^{dT_{sw}} v_{cp}(t)\,dt = D \langle v_{ap}(t) \rangle_{T_{sw}} = DV_{ap}$$

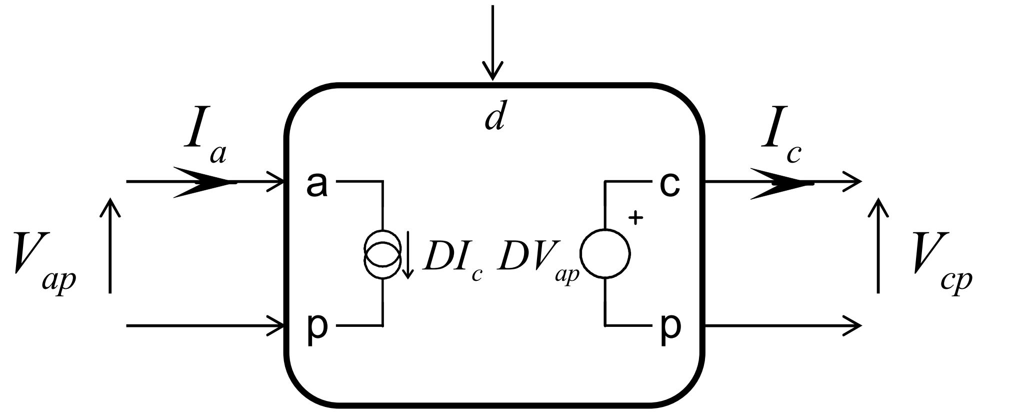
CCM

A Two-Port Representation

- We have a link between input and output variables



- It can further be illustrated with current and voltage sources



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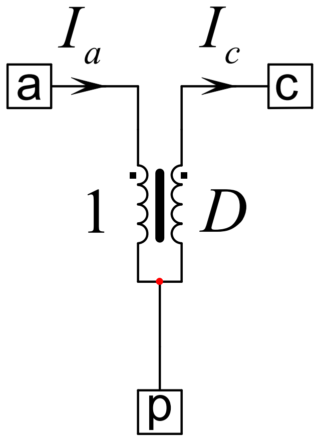
A Transformer Representation

- The PWM switch large-signal model is a dc "transformer"!

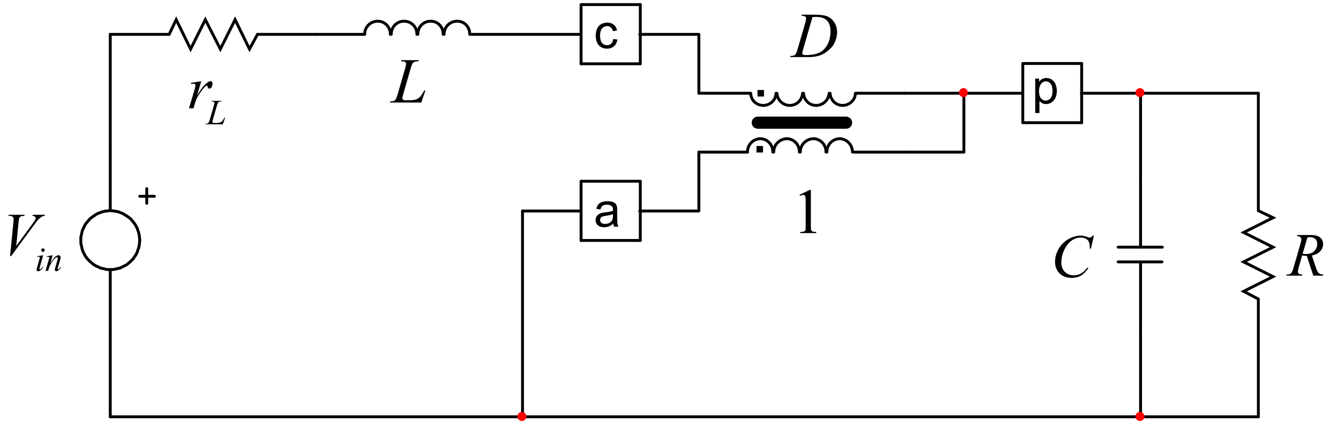


$$I_a = DI_c \qquad I_c = \frac{I_a}{D}$$

$$V_{ap} = \frac{V_{cp}}{D} \qquad V_{cp} = DV_{ap}$$

dc equations!

- It can be immediately plugged into any 2-switch converter



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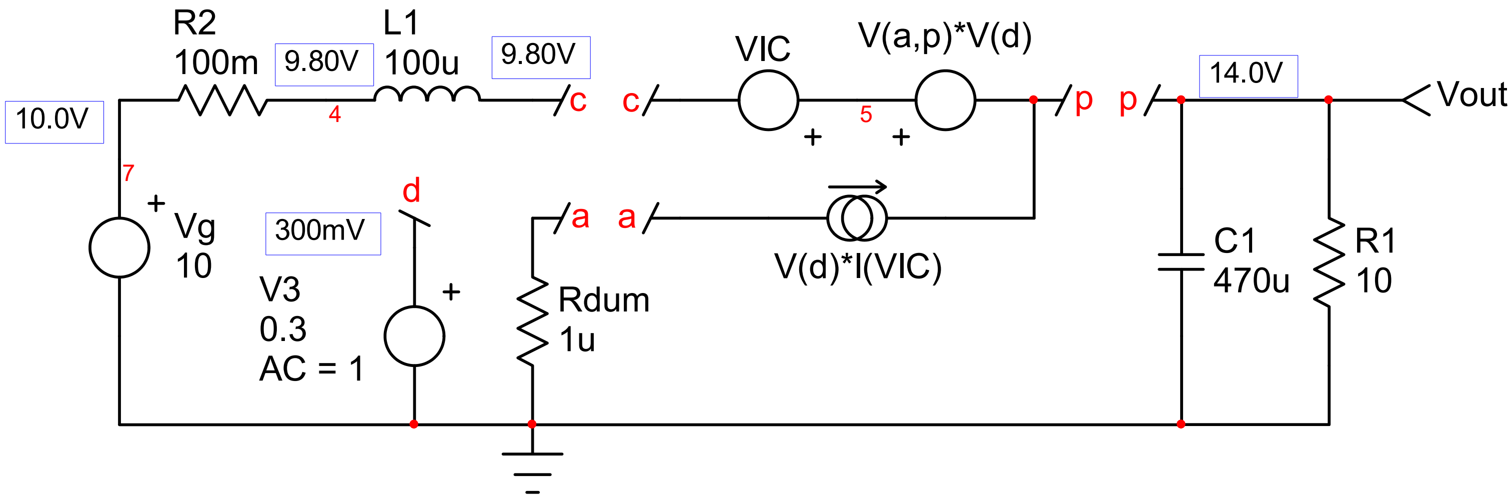
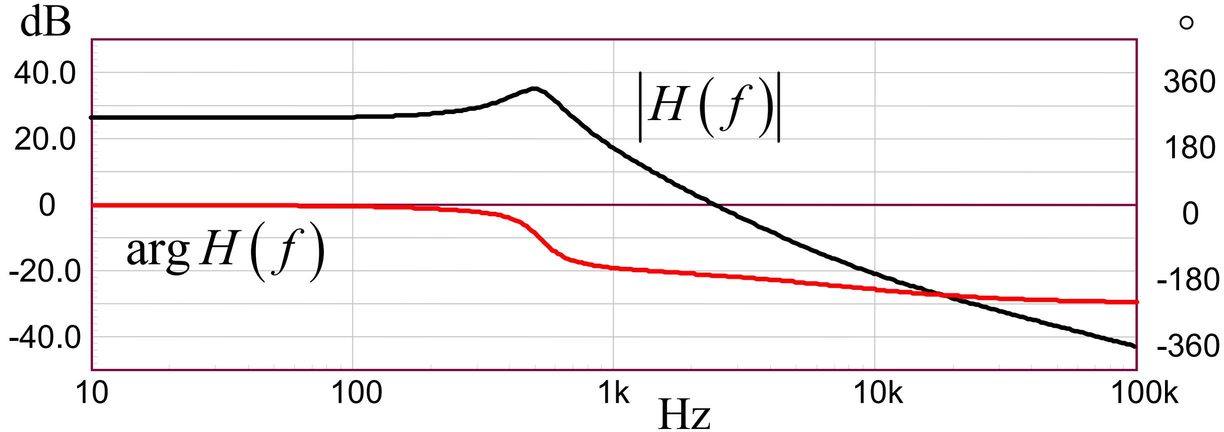
Simulate Immediately with this Model

- SPICE can get you the dc bias point



- …but also the ac response as it linearizes the circuit



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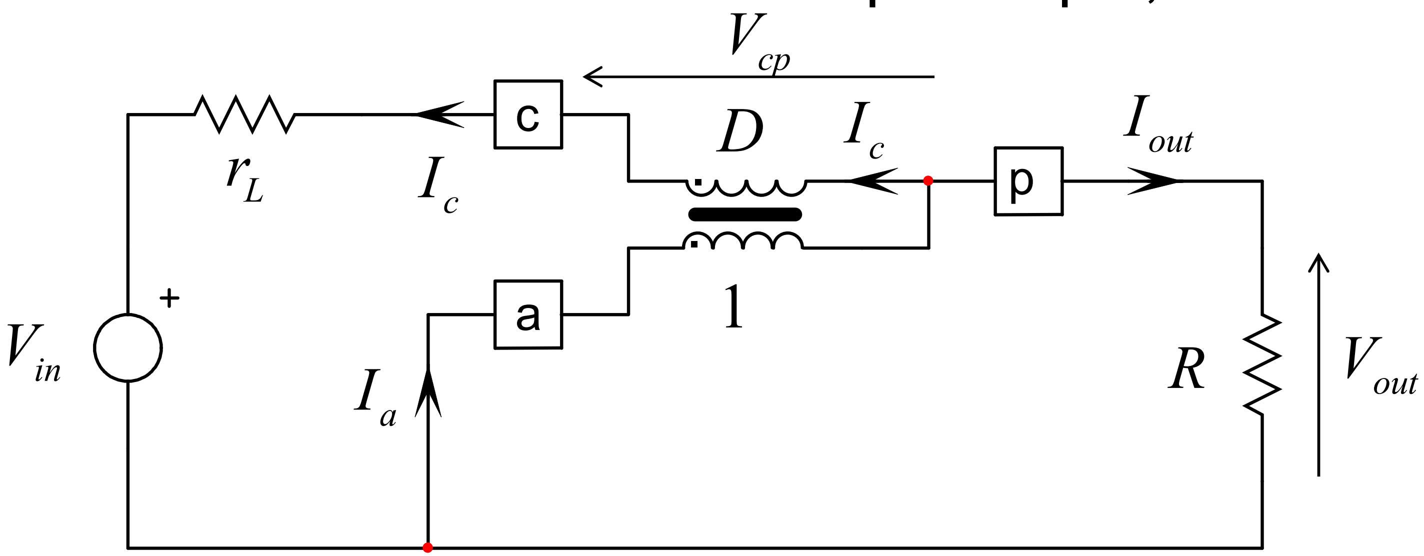
We Want Transfer Functions

- ❑ Derive the dc transfer function: open caps., short inductors



$$V_{out} = I_{out}R$$

$$V_{out} = (I_a - I_c)R$$

$$I_a = DI_c$$

$$V_{out} = I_c(D-1)R$$

$$V_{in} + r_L I_c - V_{cp} = V_{out}$$

$$V_{in} + r_L I_c + DV_{out} = V_{out}$$

$$I_c = \frac{V_{out}(1-D) - V_{in}}{r_L}$$

$$\frac{V_{out}}{V_{in}} = \frac{1}{(1-D) - \frac{r_L}{(D-1)R}}$$

$$\frac{V_{out}}{V_{in}} = \frac{1}{D'}\frac{1}{1+\frac{r_L}{RD'^2}}$$

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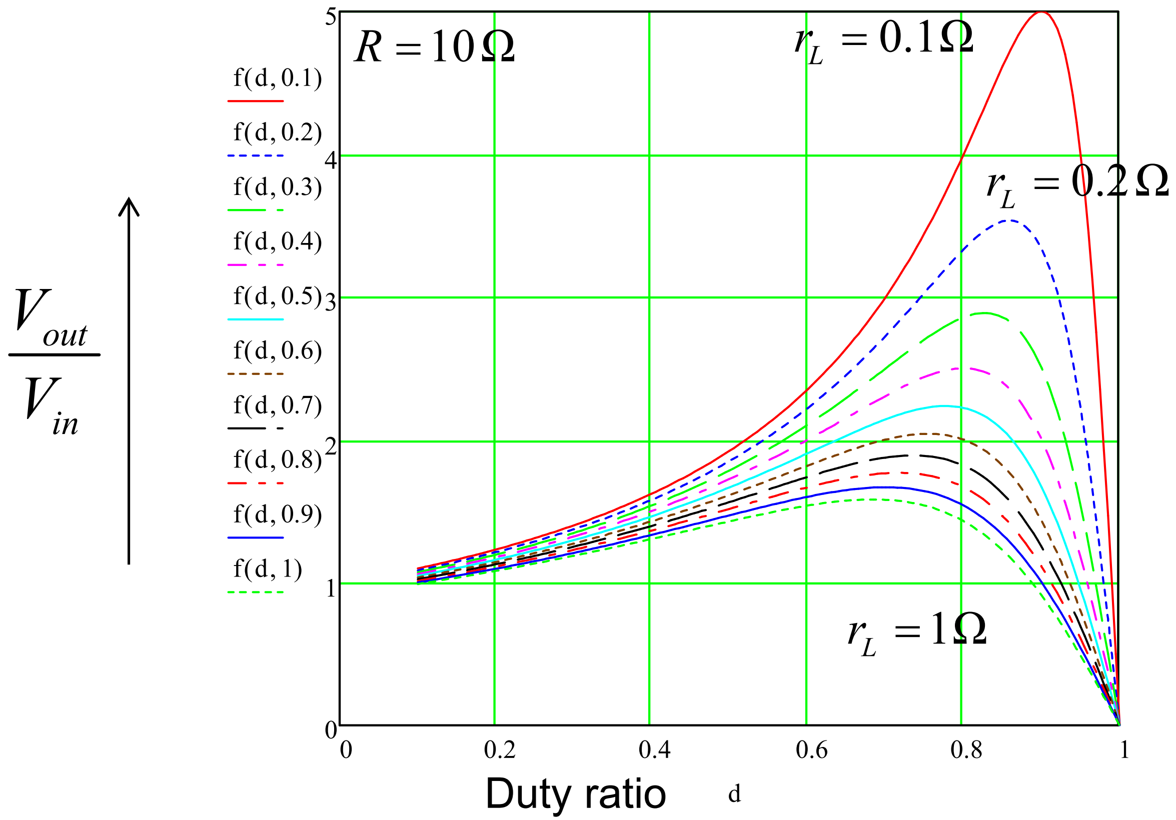
Plotting Transfer Functions

❑ Plot the lossy boost transfer function in a snapshot



❑ Above a certain conversion ratio, latch-up occurs

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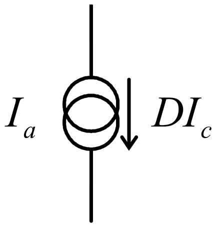
- ❑ Peak Current Mode Control in Large Signal
- ❑ The CCM CM Small-Signal PWM Switch Model
- ❑ The DCM CM Small-Signal PWM Switch Model
- ❑ The PWM Switch in Boundary Mode

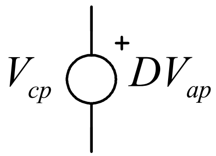
A Small-Signal Model

- ❑ We need a small-signal version to get the ac response
- ❑ We can apply the linearization technique we learned



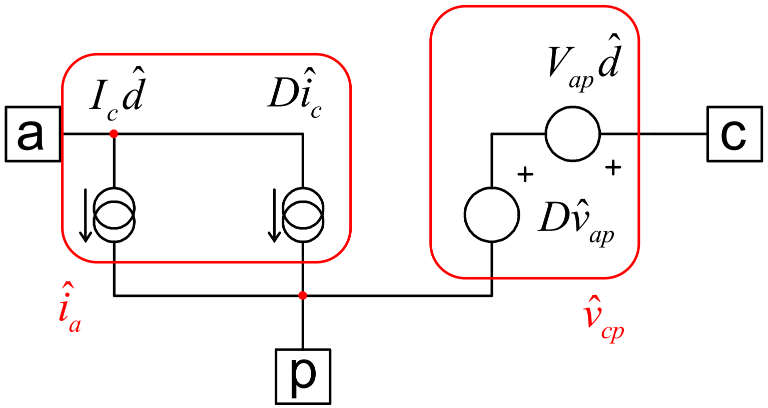
2 variables

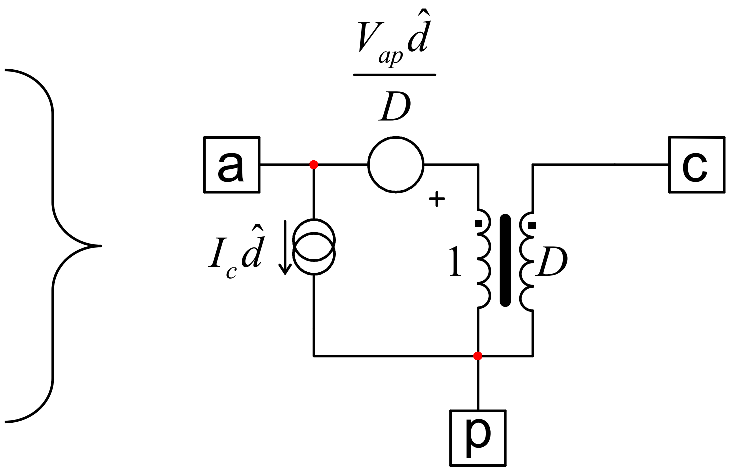
$$\hat{i}_a = \frac{\partial f(D, I_c)}{\partial D}\hat{d} + \frac{\partial f(D, I_c)}{\partial I_c}\hat{i}_c \Rightarrow \hat{i}_a = I_c\hat{d} + D\hat{i}_c$$



2 variables

$$\hat{v}_{cp} = \frac{\partial f(D, V_{ap})}{\partial D}\hat{d} + \frac{\partial f(D, V_{ap})}{\partial V_{ap}}\hat{v}_{ap} \Rightarrow \hat{v}_{cp} = V_{ap}\hat{d} + D\hat{v}_{ap}$$



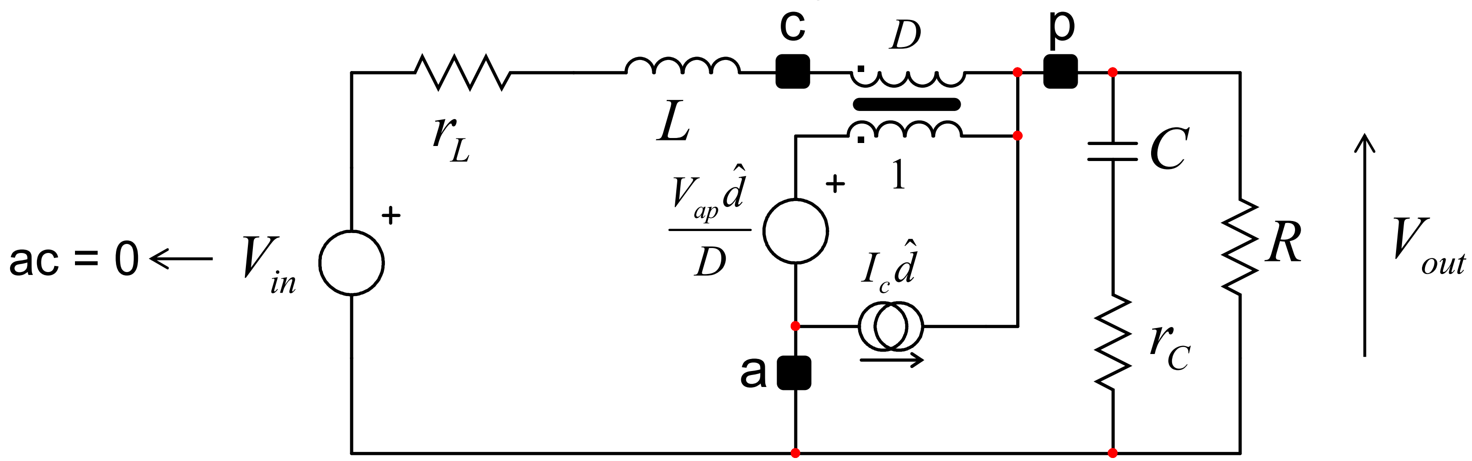


Small-signal model

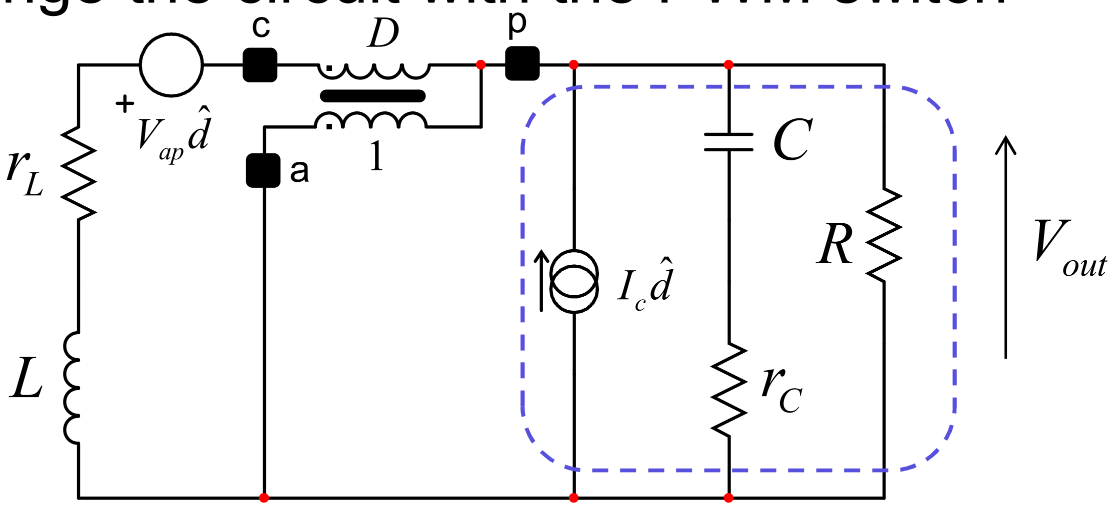
CCM

Plug this Small-Signal Model in the Boost

- You now have a completely linear and continuous model

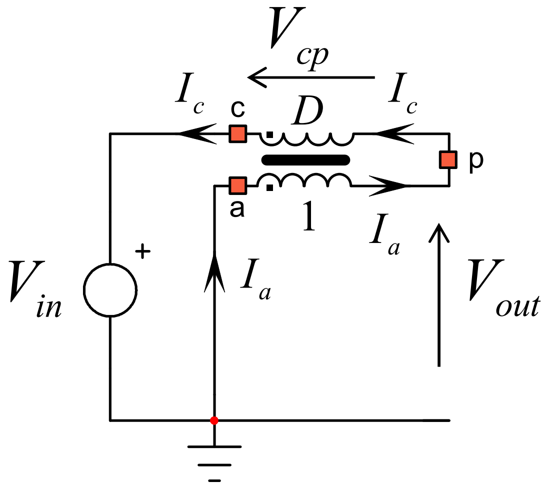


- Re-arrange the circuit with the PWM switch



Reflect on the Other Side if Needed

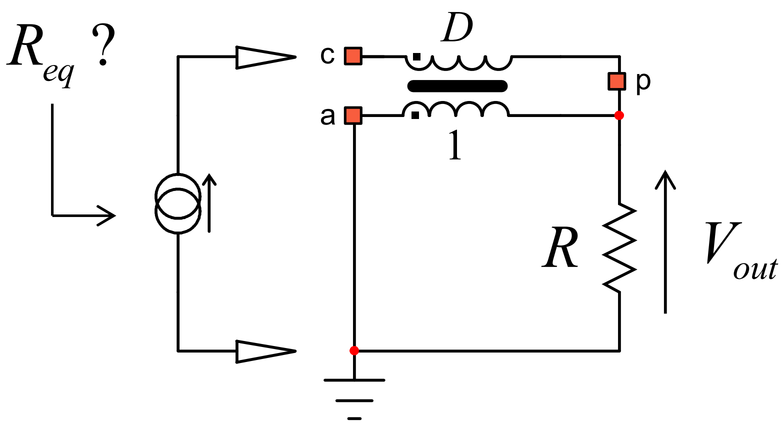
- Write dc equations to define the dc transfer function



$$V_{in} - V_{cp} = V_{out} \qquad V_{cp} = -DV_{out}$$

$$V_{in} + DV_{out} = V_{out} \qquad \frac{V_{out}}{V_{in}} = \frac{1}{1-D} = \frac{1}{D'}$$

- Apply a similar technique to reflect impedances



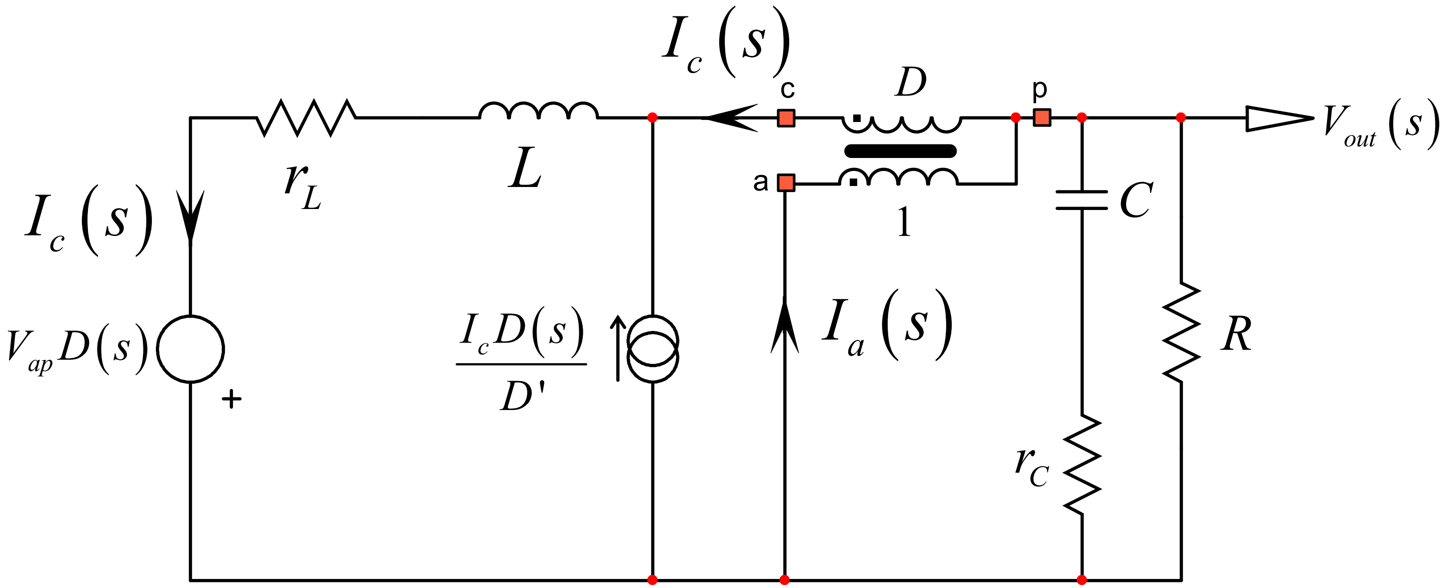
$$P_{out} = P_{in} = \frac{V_{out}^{2}}{R} = \frac{V_{in}^{2}}{R_{eq}}$$

$$\frac{V_{out}^{2}}{R} = \frac{V_{out}^{2} D'^{2}}{R_{eq}}$$

$$\left. \right\} \; R_{eq} = RD'^{2}$$

Simplify the Circuit and Solve it!

- The circuit can now be updated with Laplace notations

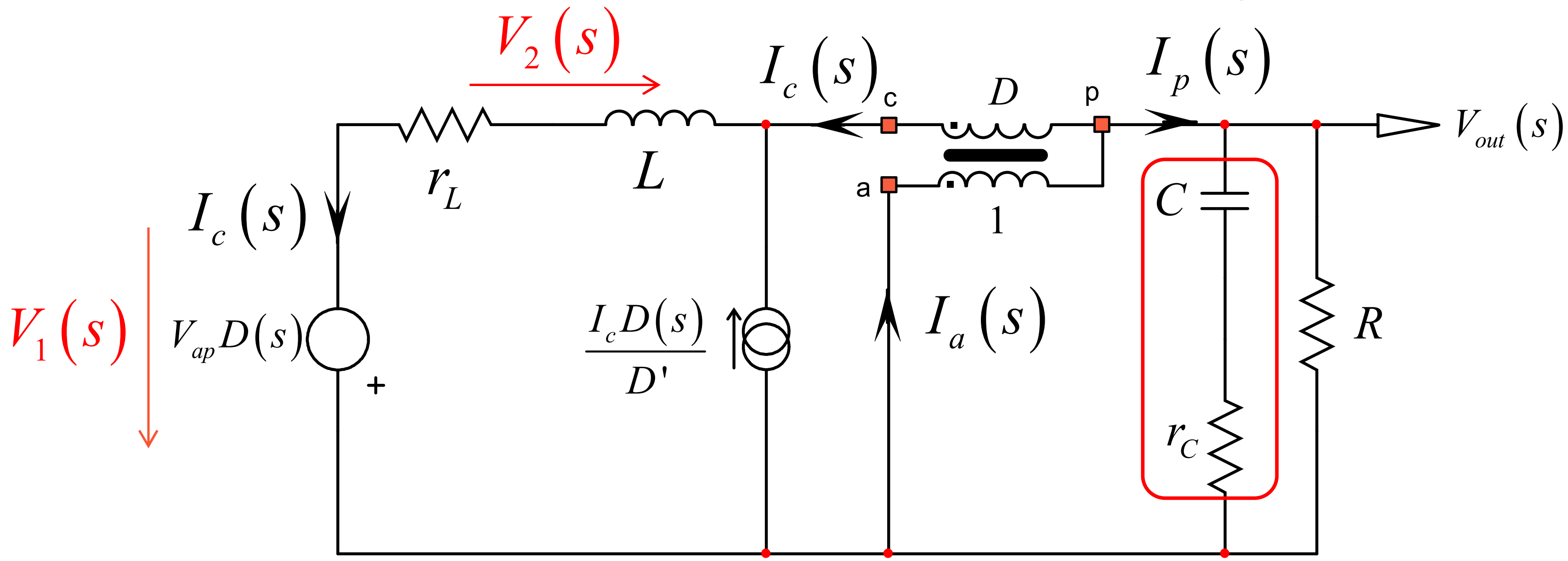


- We have two storage elements, it is a second-order system

$$H(s) = \frac{V_{out}(s)}{D(s)} = H_0 \frac{N(s)}{D(s)} = H_0 \frac{\left(1+s_{z_1}\right)\left(1+s_{z_2}\right)\left(1+...\right)}{1+\frac{s}{\omega_0 Q}+\left(\frac{s}{\omega_0}\right)^2}$$

Identify the Zeros

- A zero prevents the excitation from reaching the output



- No response means: $r_C C$ is a short or $\hat{i}_p = 0$

$$\left.\begin{aligned} V_1(s) - V_2(s) &= 0 \\ r_C + \frac{1}{sC} &= 0 \end{aligned}\right\} \quad \begin{aligned} V_{ap}D(s) - I_c(s)(sL + r_L) &= 0 \\ \frac{sr_C C + 1}{sC} &= 0 \end{aligned}$$

Two zeros

Identify the Zeros

- The first zero is straightforward

$$sr_C C + 1 = 0 \quad \Rightarrow \quad s_{z_1} = -\frac{1}{r_C C}$$ This is a LHP zero

- We know $I_a(s)$ from the small-signal model

$$I_a(s) = DI_c(s) + I_c D(s)$$

- Since $I_p(s) = 0$, we have $I_a(s) = I_c(s) \longrightarrow I_c(s) = DI_c(s) + I_c D(s)$

- We can rearrange the equations

$$V_{ap} D(s) - I_c(s)(sL + r_L) = 0 \longrightarrow I_c(s) = \frac{V_{ap} D(s)}{sL + r_L}$$

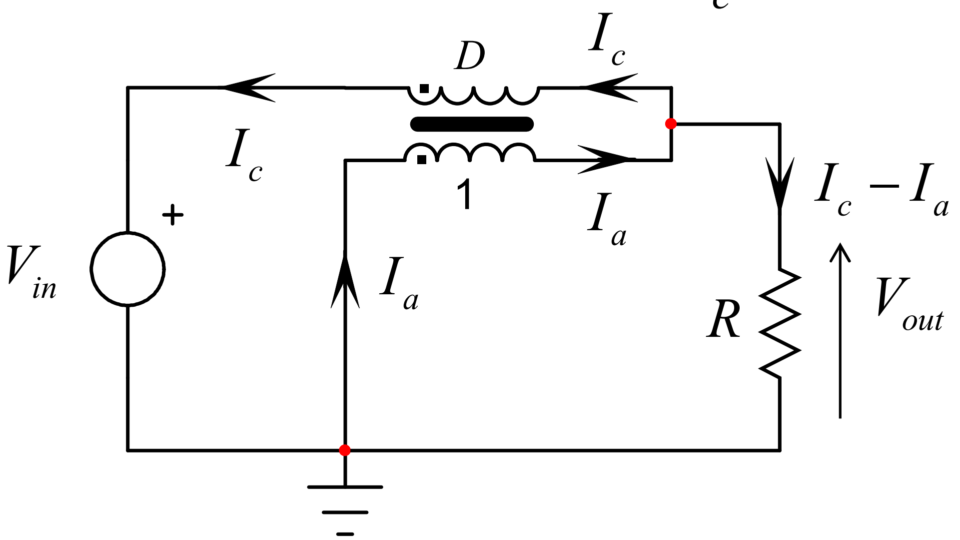
$$I_c(s) = DI_c(s) + I_c D(s) \longrightarrow I_c(s)(1 - D) = I_c D(s) \longrightarrow I_c(s) = \frac{I_c D(s)}{D'}$$

Identify the Zeros

- ❑ Equate and solve for s

$$\frac{I_c D(s)}{D'} = \frac{V_{ap} D(s)}{sL + r_L} \longrightarrow s_{z_1} = \frac{D'V_{ap} - I_c r_L}{I_c L} = \frac{1}{L}\left(D'\frac{V_{ap}}{I_c} - r_L\right)$$

- ❑ What is the value of I_c?



$$P_{in} = P_{out}$$

$$-I_c V_{in} = (I_c - I_a) V_{out} = I_{out} V_{out}$$

$$\frac{V_{out}}{V_{in}} = \frac{1}{D'}$$

$$\rightarrow I_c = -\frac{I_{out}}{D'}$$

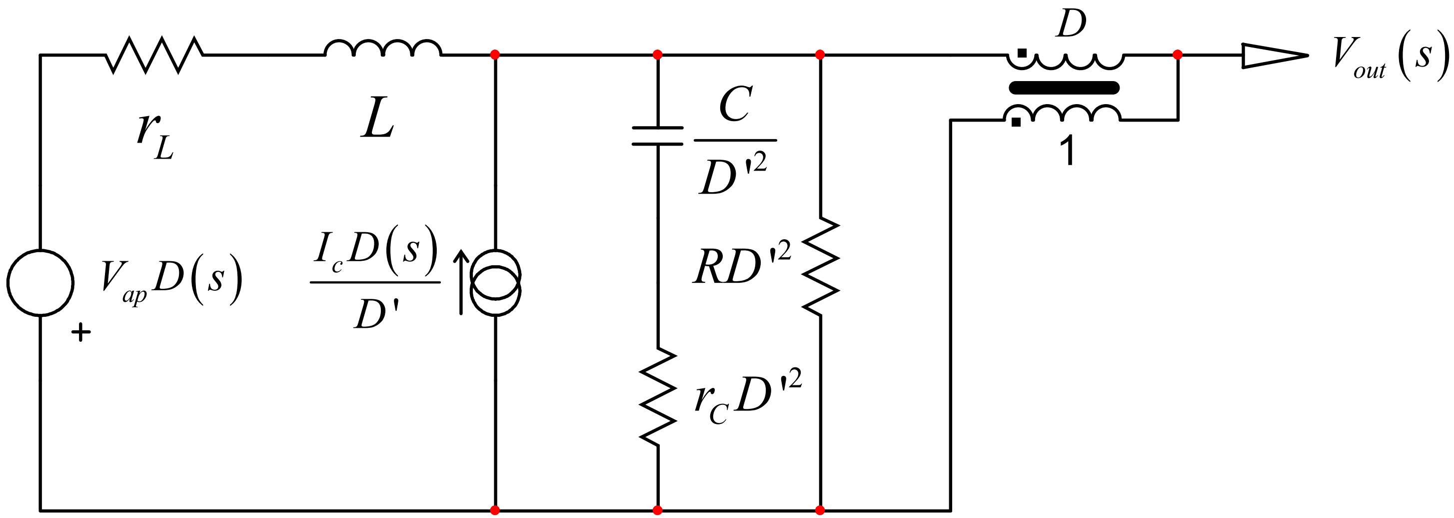
$$\rightarrow V_{ap} = -V_{out}$$

$$\frac{V_{ap}}{I_c} = D'R$$

$$s_{z_1} = \frac{1}{L}\left(D'^2 R - r_L\right)$$ This is a RHP zero

Identify the Poles

- ❑ Let's reflect the load and capacitor on the other side

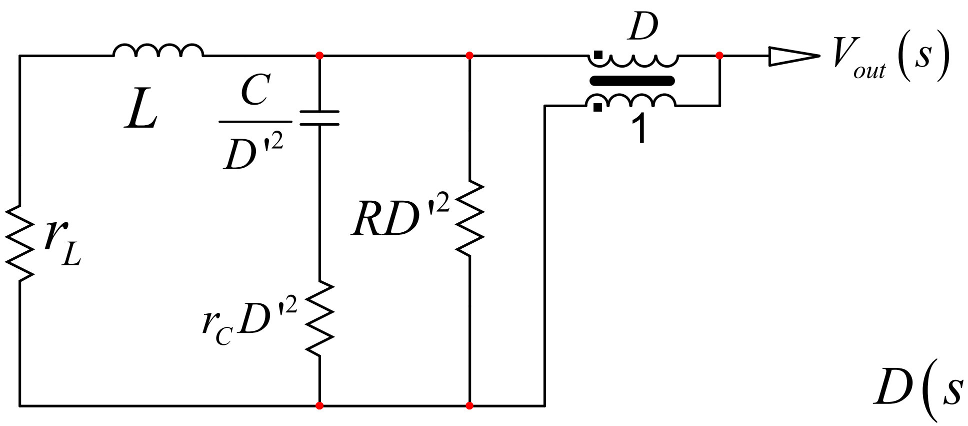


- ❑ The denominator is solely dependent on the structure

➢ It is independent from the excitation: set it to zero!
➢ Identify the time constants

Identify the Poles

- We reduce the excitation to zero: $D(s)=0$



This is a two-storage elements network

$$D(s)=1+a_1 s+a_2 s^2=1+\frac{s}{\omega_0 Q}+\left(\frac{s}{\omega_0}\right)^2$$

- D must be dimensionless thus: $a_1 \equiv (Hz)^{-1}$ $a_2 \equiv (Hz)^{-2}$

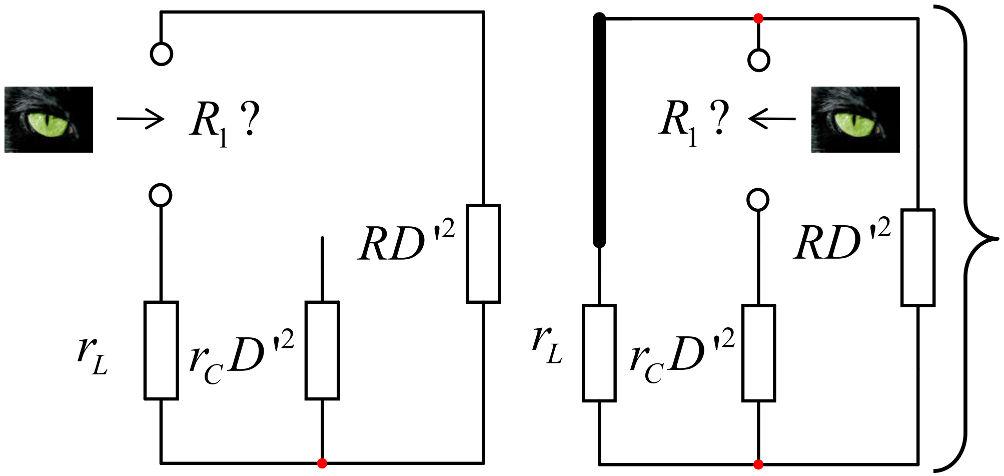
- The two possible terms for a_1 are $\tau_1+\tau_2$
- The two possible terms for a_2 are $\tau_1 \tau'_2$, $\tau'_1 \tau_2$

$$\tau=\frac{L}{R} \text{ or } \tau=RC$$

B. Erickson, "The *n* Extra Element Theorem", http://ecee.colorado.edu/copec/publications.php

Identify the Poles

- For a_1 look at the resistance R driving L and C
  - Look at the driving impedance at L while C is in its dc state
  - Look at the driving impedance at C while L is in its dc state



$$\ldots s\left(\underbrace{\frac{L}{r_L + RD'^2}}_{\tau_1} + \underbrace{\frac{C}{D'^2}\left[\left(r_L \parallel RD'^2\right) + r_C D'^2\right]}_{\tau_2}\right)$$

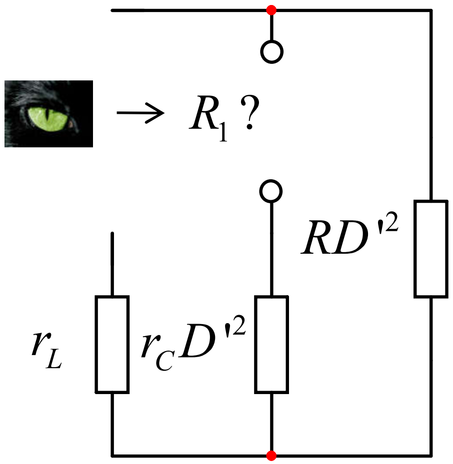
$$R_1 = r_L + RD'^2 \qquad R_1 = \left(r_L \parallel RD'^2\right) + r_C D'^2$$

Identify the Poles

- ❑ how τ_1 (involving L) combines with τ'_2 (involving C)?
- ❑ how τ_2 (involving C) combines with τ'_1 (involving L)?

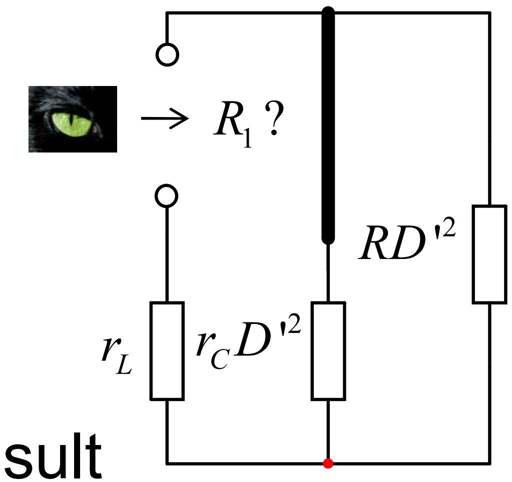
$\Big\}\ a_2$

- ➢ Look at the driving impedance at C while L is in its HF state
- ➢ Look at the driving impedance at L while C is in its HF state



If we chose

$$\tau_1 = \frac{L}{r_L + RD'^2}$$



If we chose

$$\tau_2 = \frac{C}{D'^2}\left[\left(r_L \,\|\, RD'^2\right) + r_C D'^2\right]$$

$$\tau'_2 = \frac{C}{\cancel{D'^2}}\cancel{D'^2}\left(r_c + R\right)$$

Same result ⟺

$$\tau_1 \tau'_2 = \tau'_1 \tau_2$$

$$\tau'_1 = \frac{L}{r_L + \left(RD'^2 \,\|\, r_C D'^2\right)}$$

Identify the Poles

- We have our denominator!

$$D(s) = 1 + s\left(\frac{L}{r_L + RD'^2} + C\left[\left(\frac{r_L R}{r_L + RD'^2}\right) + r_C\right]\right) + s^2\left(LC\frac{(r_C + R)}{r_L + RD'^2}\right)$$

- We can identify the terms

$$\omega_0 = \frac{1}{\sqrt{LC}}\sqrt{\frac{r_L + RD'^2}{r_C + R}} \qquad Q \approx \frac{\omega_0}{\frac{r_L}{L} + \frac{1}{C(r_C + R)}}$$

- The dc term H_0 in the transfer function is:

$$H_0 = \frac{dV_{out}(D)}{dD} \qquad V_{out}(D) \approx \frac{V_{in}}{1-D} \longrightarrow H_0 = \frac{V_{in}}{(D-1)^2} = \frac{V_{in}}{D'^2}$$

The CCM Boost Transfer Function

- The final transfer function can be written as:

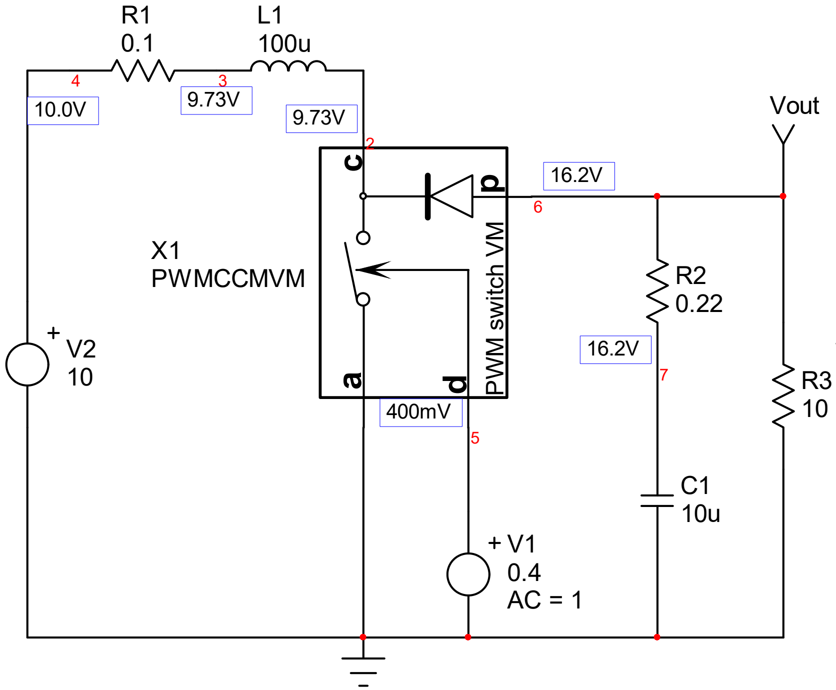
$$\frac{V_{out}(s)}{D(s)} = H_0 \frac{\left(1+\frac{s}{\omega_{z_1}}\right)\left(1-\frac{s}{\omega_{z_2}}\right)}{1+\frac{s}{\omega_0 Q}+\left(\frac{s}{\omega_0}\right)^2} \qquad \omega_{z_1} = \frac{1}{r_C C} \qquad \omega_{z_2} = \frac{1}{L}\left(D'^2 R - r_L\right)$$

$$\omega_0 = \frac{1}{\sqrt{LC}}\sqrt{\frac{r_L + RD'^2}{r_C + R}} \qquad Q \approx \frac{\omega_0}{\frac{r_L}{L}+\frac{1}{C\left(r_C+R\right)}} \qquad H_0 \approx \frac{V_{in}}{D'^2}$$

- Time to check the ac response versus simulation!

Checking Ac Responses

❑ Capture a simple boost converter schematic



$V_{in} := 10V \qquad D := 0.4$

$V_{out} := \frac{V_{in}}{1 - D} = 16.667V \qquad D_a := 1 - D = 0.6$

$L_1 := 100\mu H \qquad C_1 := 10\mu F \qquad r_L := 0.1\Omega \qquad r_C := 0.22\Omega \qquad R_1 := 10\Omega$

$\omega_{z1} := \frac{1}{r_C \cdot C_1} \qquad \omega_{z2} := \frac{1}{L_1} \cdot \left(D_a^{\ 2} \cdot R_1 - r_L\right) \qquad f_1 := \frac{\omega_{z1}}{2 \cdot \pi} = 72.343kHz$ ESR zero

$\omega_0 := \frac{1}{\sqrt{L_1 \cdot C_1}} \cdot \sqrt{\frac{r_L + D_a^{\ 2} \cdot R_1}{r_C + R_1}} \qquad f_2 := \frac{\omega_{z2}}{2 \cdot \pi} = 5.57 \times 10^3 \, Hz$ RHP zero

$f_0 := \frac{\omega_0}{2 \cdot \pi} = 3.028 \times 10^3 \, Hz$

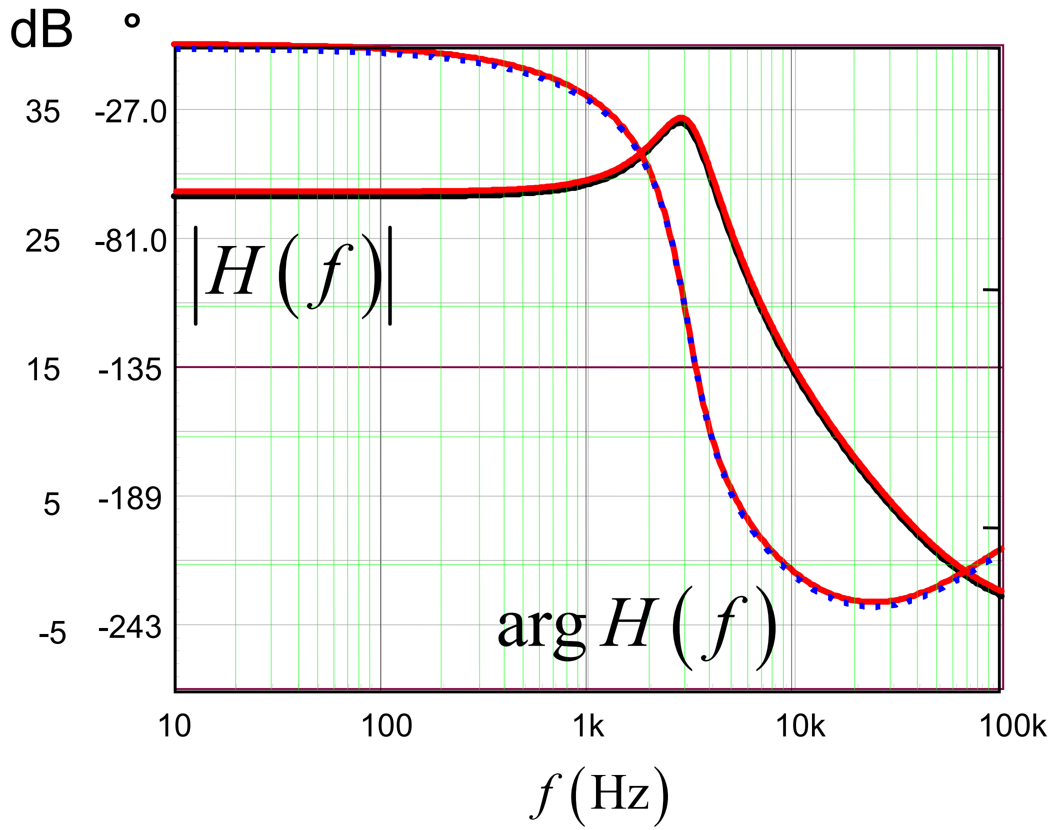
$Q := \frac{1}{\omega_0 \cdot \left[C_1 \cdot \left(r_C + \frac{R_1 \cdot r_L}{R_1 \cdot D_a^{\ 2} + r_L}\right) + \frac{L_1}{R_1 \cdot D_a^{\ 2} + r_L}\right]} = 1.646 \qquad H_0 := \frac{V_{in}}{D_a^{\ 2}}$

$$H_1(s) := H_0 \cdot \frac{\left(1 + \frac{s}{\omega_{z1}}\right) \cdot \left(1 - \frac{s}{\omega_{z2}}\right)}{1 + \frac{s}{\omega_0 \cdot Q} + \left(\frac{s}{\omega_0}\right)^2}$$

❑ Check the result versus a Mathcad® spreadsheet

Checking the Ac Responses

- ❑ The curves must perfectly superimpose



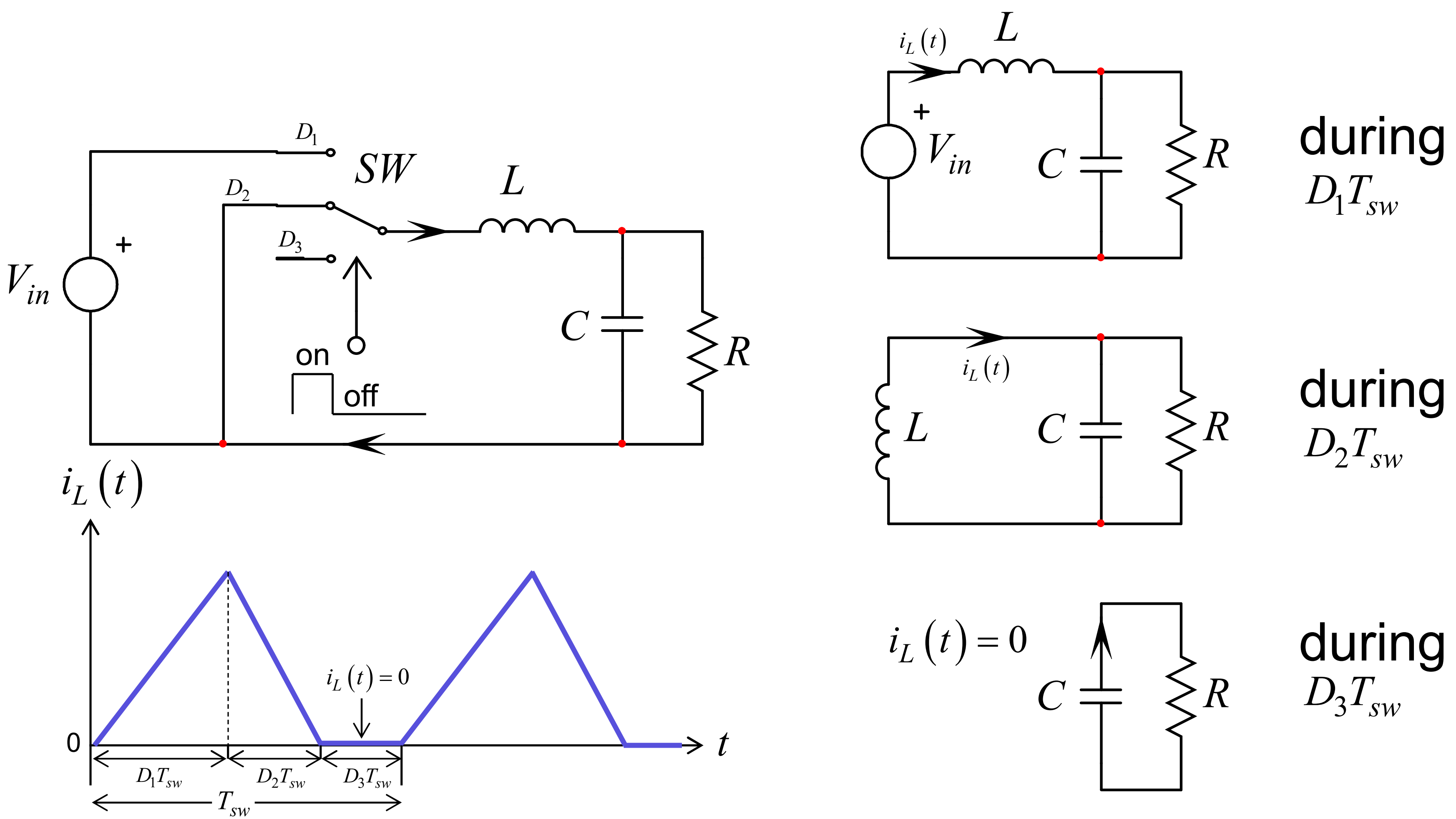
- ❑ If not, go back to the sheet and fix the equations!

Agenda

- ❑ Linear and Non-Linear Functions
- ❑ What is a Small-Signal Model?
- ❑ Fast Analytical Techniques at Work
- ❑ From a Switched to Linearized Model
- ❑ The CCM VM Small-Signal PWM Switch Model
- ❑ **The DCM VM Small-Signal PWM Switch Model**
- ❑ Peak Current Mode Control in Large Signal
- ❑ The CCM CM Small-Signal PWM Switch Model
- ❑ The DCM CM Small-Signal PWM Switch Model
- ❑ The PWM Switch in Boundary Mode

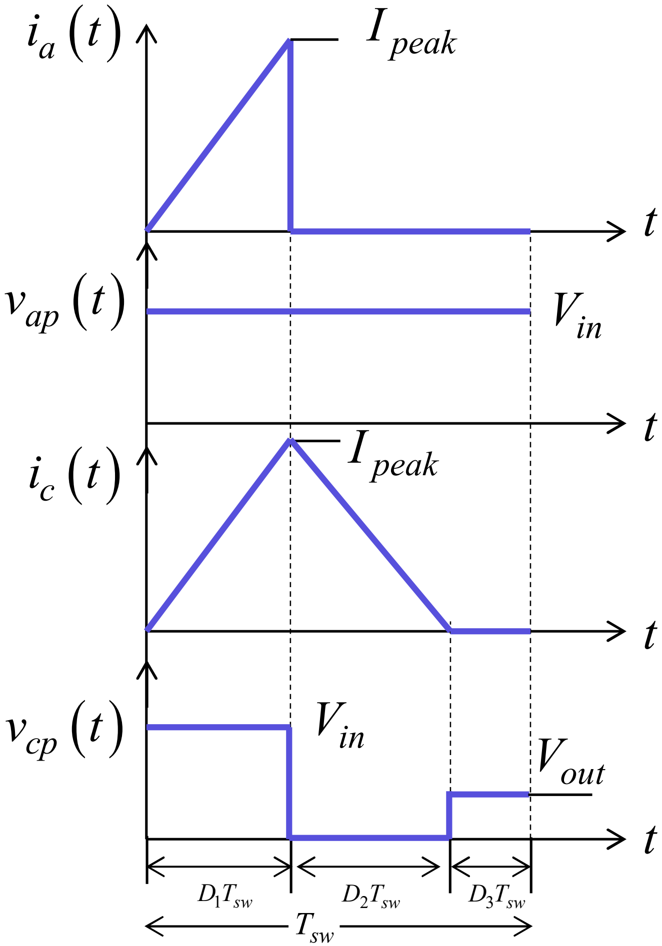
The Discontinuous Case

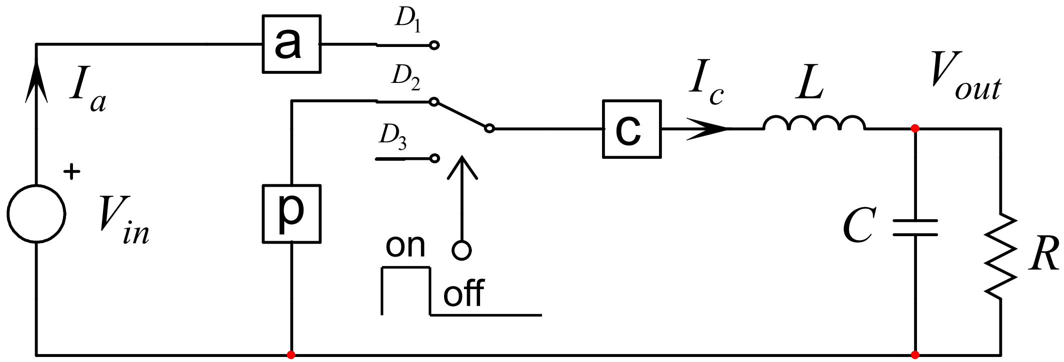
- In DCM, a third timing event exists when $i_L(t) = 0$



The Same Configuration as in CCM

❑ Draw the waveforms in the "common passive" configuration





❑ Average the waveforms:

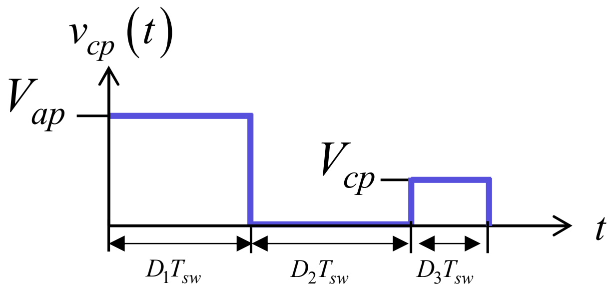
$$I_a = \frac{I_{peak}}{2} D_1$$

$$I_c = \frac{I_{peak}}{2} D_1 + \frac{I_{peak}}{2} D_2 = \frac{I_{peak}}{2}\left(D_1 + D_2\right)$$

$$I_c = \frac{2I_a}{D_1} \frac{D_1 + D_2}{2} = I_a \frac{D_1 + D_2}{D_1}$$

Derive V_{cp} to Unveil the New Model

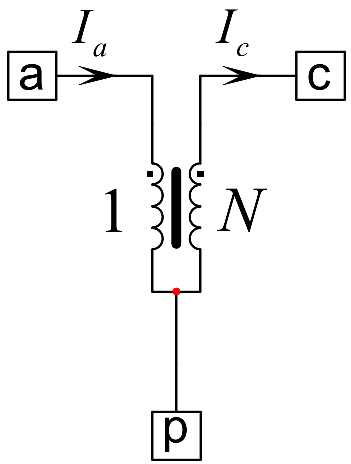
- The addition of the third event complicates the equations



$$V_{cp} = V_{ap}D_1 + V_{cp}D_3 \qquad D_1 + D_2 + D_3 = 1$$

$$V_{cp} = V_{ap}D_1 + V_{cp}\left(1 - D_1 - D_2\right)$$

$$V_{cp} = V_{ap}\frac{D_1}{D_1 + D_2}$$



$$I_a = NI_c \qquad I_c = \frac{I_a}{N}$$

$$V_{ap} = \frac{V_{cp}}{N} \qquad V_{cp} = NV_{ap}$$

Control input

$$N = \frac{D_1}{D_1 + D_2}$$

$\rightarrow f(D_1)$

Finally, Get the D_2 Value

- In DCM the inductor average voltage per cycle is always 0

$$\longrightarrow \quad V_{cp} = V_{out}$$

- What is the averaged inductor peak current?

$$\left.\begin{aligned} I_{peak} &= \frac{\langle v_L(t)\rangle_{D_1T_{sw}}}{L} D_1T_{sw} \\ \langle v_L(t)\rangle_{D_1T_{sw}} &= V_{ac} \end{aligned}\right\} \quad V_{ac} = L\frac{\boxed{I_{peak}}}{D_1T_{sw}}$$

- The peak current uses a previous expression

$$I_c = \frac{I_{peak}}{2}(D_1 + D_2) \longrightarrow \boxed{I_{peak}} = \frac{2I_c}{D_1 + D_2}$$

$$\Rightarrow \quad D_2 = \frac{2LF_{sw}}{D_1}\frac{I_c}{V_{ac}} - D_1$$

CCM to DCM Auto-Toggling is Possible

- We have two equations for the DCM model

$$I_c = I_a \frac{D_1 + D_2}{D_1} \qquad V_{cp} = V_{ap} \frac{D_1}{D_1 + D_2}$$

- When D_3 shrinks and finally disappears

$$D_1 + D_2 + D_3 = 1 \longrightarrow D_2 = 1 - D_1$$

- If you substitute D_2 in the two above equations

$$I_c = I_a \frac{D_1 + (1 - D_1)}{D_1} \longrightarrow I_a = D_1 I_c$$

$$V_{cp} = V_{ap} \frac{D_1}{D_1 + (1 - D_1)} \longrightarrow V_{cp} = D_1 V_{ap}$$

CCM equations

Clamp the Equation to Auto Toggle

- D_2 in DCM is smaller than D_2 in CCM

$$D_{2,DCM} = 1 - D_1 - D_3 \qquad D_{2,CCM} = 1 - D_1$$

- Clamping D_2 equation to $1 - D_1$ offers auto-toggling

IF $\frac{2LF_{sw}}{D_1}\frac{I_c}{V_{ac}} - D_1 > 1 - D_1$ THEN $D_2 = 1 - D_1$ (CCM)

ELSE $D_2 = \frac{2LF_{sw}}{D_1}\frac{I_c}{V_{ac}} - D_1$ (DCM)

- SPICE offers several means to clamp

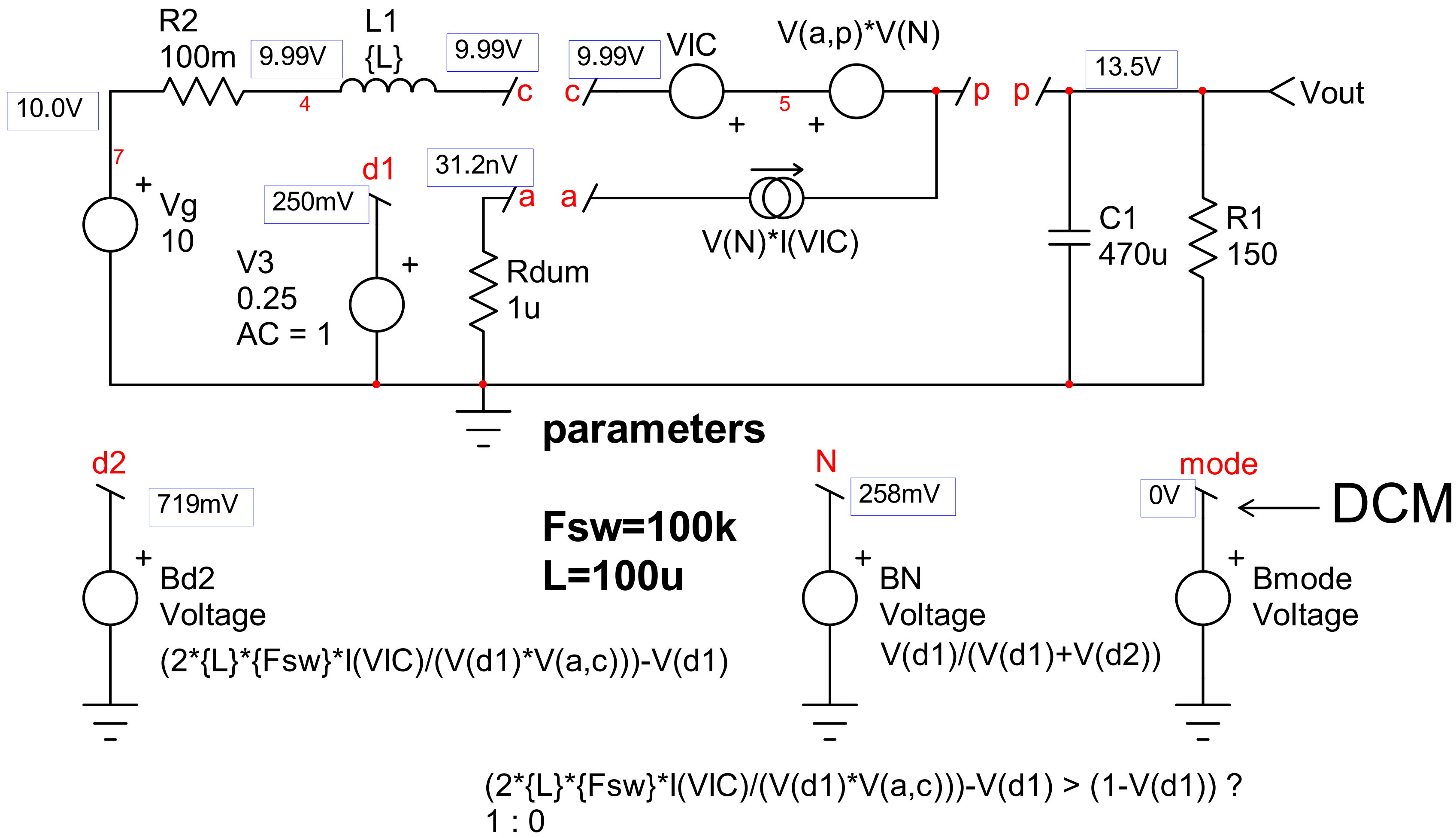
PSpice
```
Ed2 d2 0 value = { IF ( ((2*{L}*{Fs}*I(VM)/(V(dc)*V(a,c)+1u)) - V(dc))>(1-V(D)),1-V(D),
+ 2*{L}*{Fs}*I(VM)/(V(dc)*V(a,c)+1u)) - V(dc) ) }
```

IsSpice
```
Bd2 d2 0 V =((2*{L}*{Fs}*I(VM)/(V(dc)*V(a,c)+1u)) - V(dc))>(1-V(D)) ? 1-V(D) :
+ (2*{L}*{Fs}*I(VM)/(V(dc)*V(a,c)+1u)) - V(dc)
```

C. Basso,"SMPS: SPICE Simulations and Practical Designs", McGraw-Hill 2008

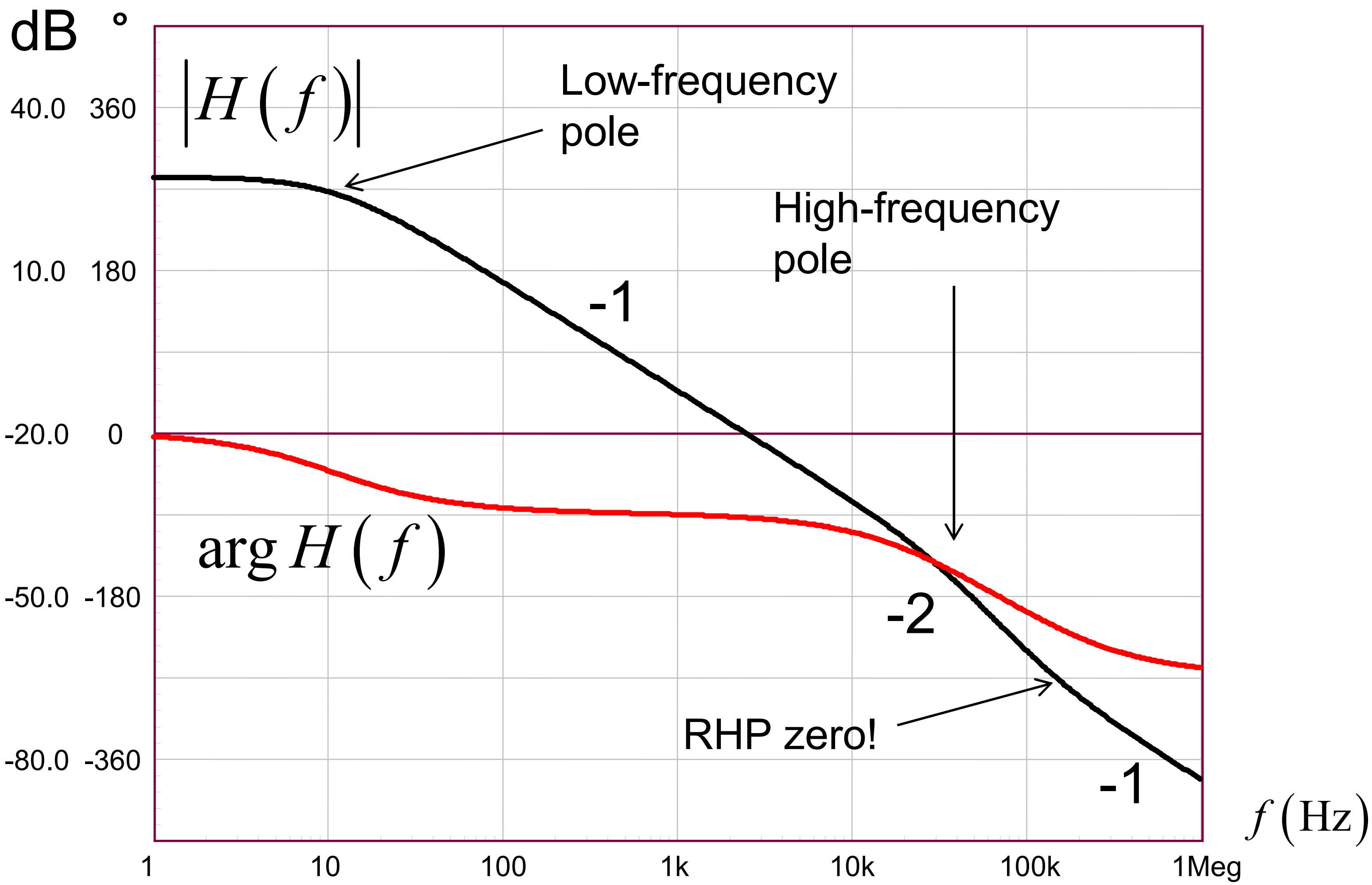
Run a Simulation Immediately!

- Re-use the CCM boost example and plug the DCM model



Ac Response in a Snapshot

- The ac response is that of a second-order system!



Calculate the Dc Operating Point

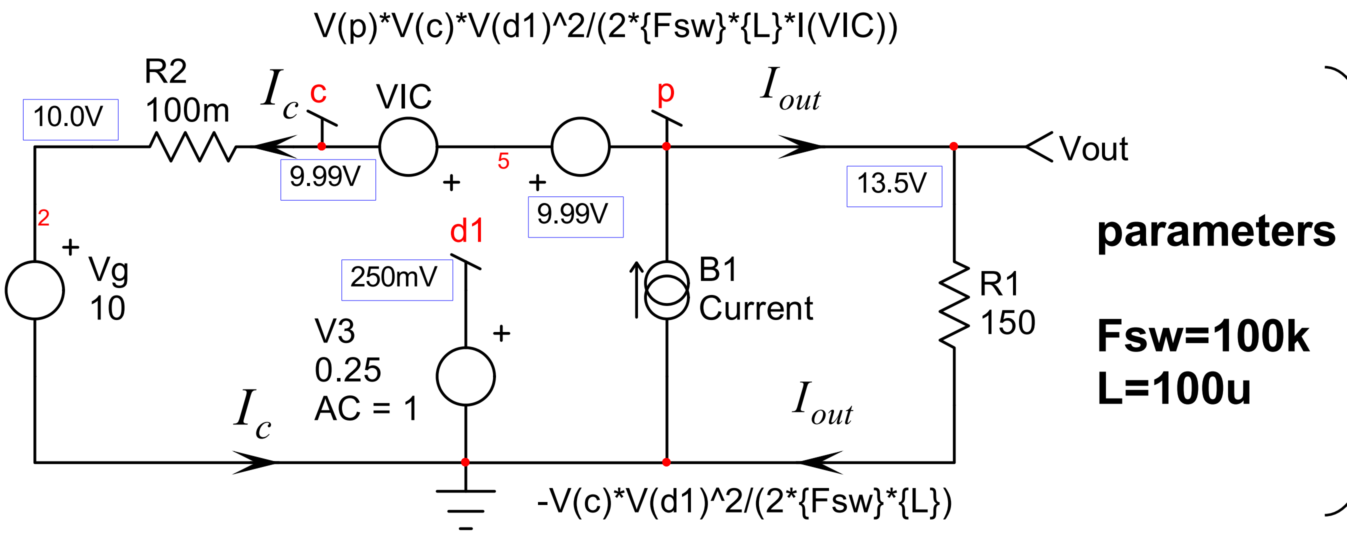
- Let's rework the DCM model to fit that of the CCM

$$N = \frac{D_1}{D_1 + D_2}$$

$$D_2 = \frac{2LF_{sw}}{D_1}\frac{I_c}{V_{ac}} - D_1$$

$$N = \frac{V_{ac}D_1^{\ 2}}{2F_{sw}I_cL}$$

- Short inductors, open caps. and verify bias point



Equations:

$$V_c = V_{in} + R_2 I_c \qquad 1.$$

$$-\frac{V_c D^2}{2F_{sw}L} = \frac{V_{out}}{R_1} + I_c \qquad 2.$$

$$V_{(c)} = V_{out} + \frac{V_{out}V_c D^2}{2LF_{sw}I_c} \qquad 3.$$

Identify the Duty Ratio and M Expressions

- Solve I_c and V_c from the first equations:

$$\longrightarrow V_{(c)} = \frac{V_{in} - \frac{V_{out} R_2}{R_1}}{\frac{D^2 R_2}{2LF_{sw}} + 1} \approx V_{in}$$ Voltage at node c

$$R_2 = 0$$

$$\longrightarrow I_c = -\frac{R_1 V_{in} D^2 + 2LF_{sw} V_{out}}{R_1 \left(R_2 D^2 + 2F_{sw} L\right)} \approx \frac{V_{out}}{R_1}\left(-1 - \frac{V_{in} D^2 R_1}{2LV_{out} F_{sw}}\right)$$

- Substitute these variables in the third equation

$$V_{in} = \frac{2LF_{sw} V_{out}{}^2}{R_1 V_{in} D^2 + 2LF_{sw} V_{out}}$$

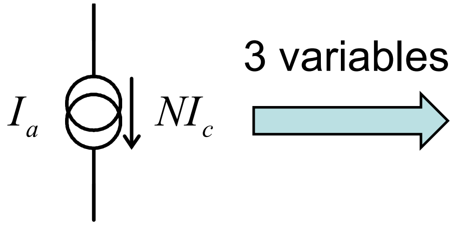
$$\underbrace{\left(\frac{V_{in}}{V_{out}}\right)^2}_{(1/M)^2} R_1 D^2 + 2LF_{sw} \underbrace{\frac{V_{in}}{V_{out}}}_{M} = 2LF_{sw}$$

$$D = \sqrt{\frac{2LF_{sw} V_{out}\left(\frac{V_{out}}{V_{in}} - 1\right)}{V_{in} R_1}} = 0.251$$ ok

$$M = \frac{1}{2}\left(1 + \sqrt{1 + \frac{2R_1 D^2}{LF_{sw}}}\right) = 1.348$$ ok

Deriving the Small-Signal Model

- Large-signal expressions require linearization
- Apply partial derivatives to the equations



$$\hat{i}_a = \frac{\partial f\left(D_1, I_c, V_{ac}\right)}{\partial D_1}\hat{d}_1 + \frac{\partial f\left(D_1, I_c, V_{ac}\right)}{\partial I_c}\hat{i}_c + \frac{\partial f\left(D_1, I_c, V_{ac}\right)}{\partial V_{ac}}\hat{v}_{ac}$$

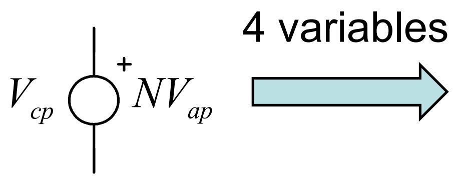
- Mathcad® automates the process easily

$$\hat{i}_a = \frac{V_{ac} D_1}{F_{sw} L}\hat{d}_1 + \frac{D_1^{\,2}}{2F_{sw} L}\hat{v}_{ac} \qquad k_1 = \frac{V_{ac} D_1}{F_{sw} L} \qquad k_2 = \frac{D_1^{\,2}}{2F_{sw} L}$$

$$\hat{i}_a = k_1 \hat{d}_1 + k_2 \hat{v}_{ac}$$

Small-Signal Sources in DCM

- Small-signal perturbations would be more difficult!



$$\hat{v}_{cp} = \frac{\partial f\left(D_1, I_c, V_{ac}, V_{ap}\right)}{\partial D_1}\hat{d}_1 + \frac{\partial f\left(D_1, I_c, V_{ac}, V_{ap}\right)}{\partial V_{ap}}\hat{v}_{ap} + \frac{\partial f\left(D_1, I_c, V_{ac}, V_{ap}\right)}{\partial I_c}\hat{i}_c + \frac{\partial f\left(D_1, I_c, V_{ac}, V_{ap}\right)}{\partial V_{ac}}\hat{v}_{ac}$$

$$\hat{v}_{cp} = \frac{V_{ap}V_{ac}D_1}{F_{sw}I_cL}\hat{d}_1 + \frac{V_{ac}D_1^2}{2F_{sw}I_cL}\hat{v}_{ap} - \frac{V_{ap}V_{ac}D_1^2}{2F_{sw}I_c^2L}\hat{i}_c + \frac{V_{ap}D_1^2}{2F_{sw}I_cL}\hat{v}_{ac}$$

$$k_3 = \frac{V_{ap}V_{ac}D_1}{F_{sw}I_cL} \quad k_4 = \frac{V_{ac}D_1^2}{2F_{sw}I_cL} \quad k_5 = -\frac{V_{ap}V_{ac}D_1^2}{2F_{sw}I_c^2L} \quad k_6 = \frac{V_{ap}D_1^2}{2F_{sw}I_cL}$$

$$\hat{v}_{cp} = k_3\hat{d}_1 + k_4\hat{v}_{ap} + k_5\hat{i}_c + k_6\hat{v}_{ac}$$

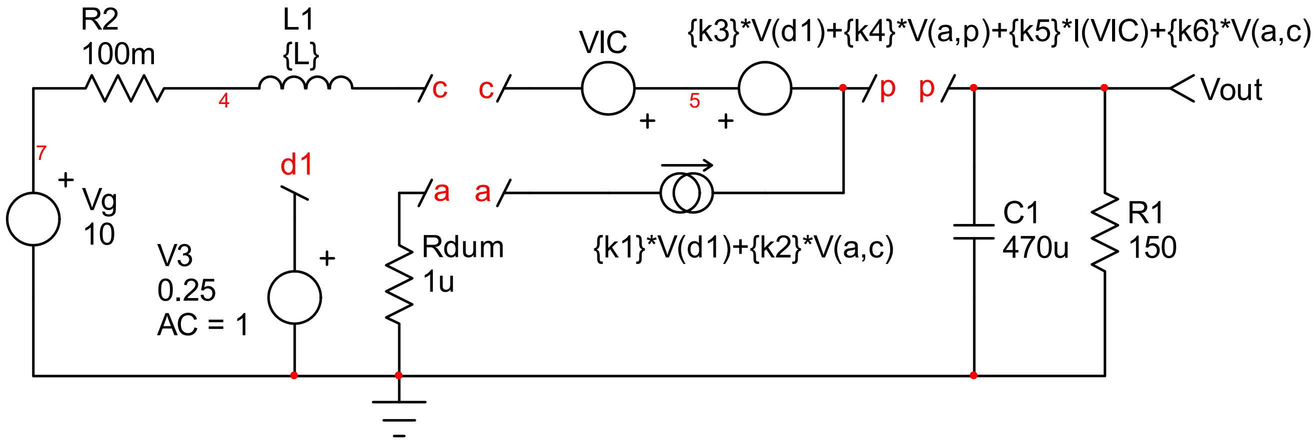
- A sanity check is recommended before proceeding

Verifying the Intermediate Step

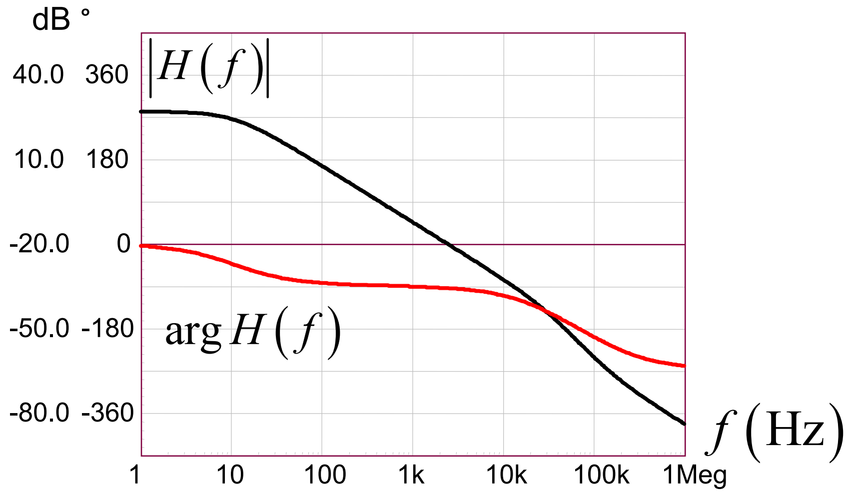
❑ A SPICE simulation will let us know if the derivation is ok

parameters

Fsw=100k
L=100u
d1=250m
Vac=-9.99
Vap=-13.5
Ia=-31.21m
Ic=-120.956m



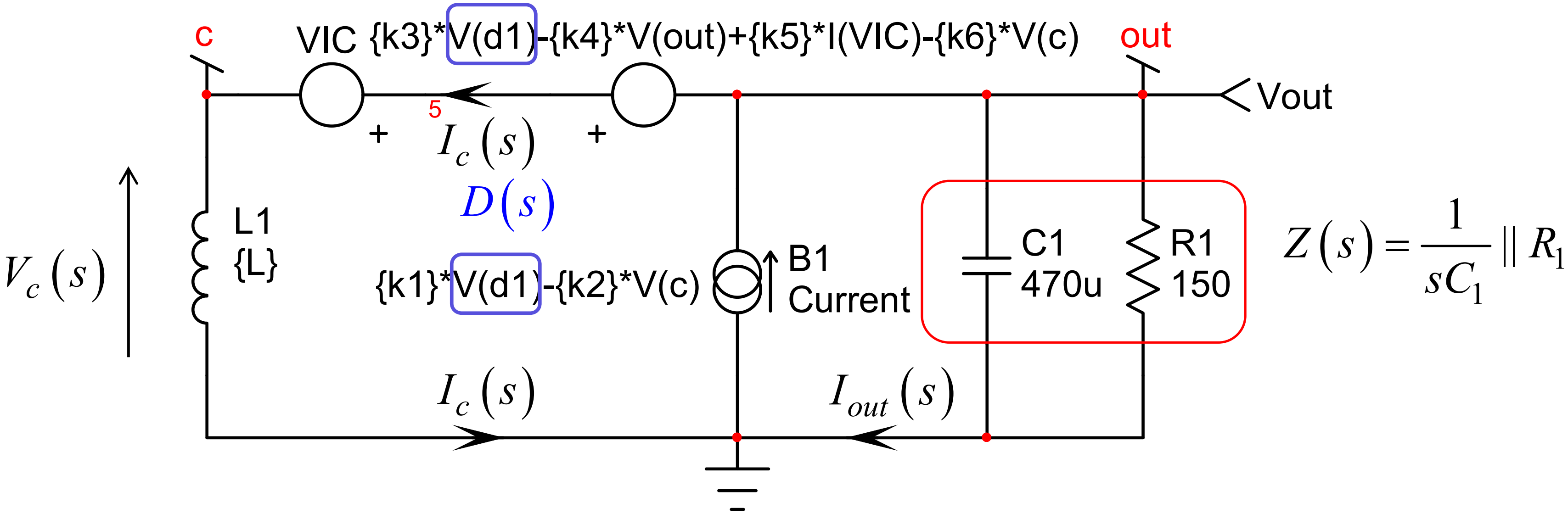
k1=Vac*d1/(Fsw*L)
k2=d1^2/(2*Fsw*L)
k3=Vac*Vap*d1/(Fsw*L*Ic)
k4=Vac*d1^2/(2*Fsw*L*Ic)
k5=-Vac*Vap*d1^2/(2*Fsw*Ic^2*L)
k6=Vap*d1^2/(2*Fsw*Ic*L)



Same results as with large-signal model!

Carefully Deriving the Expressions

- The circuit can be further simplified as $\hat{v}_{in} = 0$ is zero



- There are three equations

$$V_c(s) = I_c(s)sL$$

$$V_c(s) = V_{out}(s) + k_3 D(s) - k_4 V_{out}(s) + k_5 I_c(s) - k_6 V_c(s)$$

$$I_c(s) = k_1 D(s) - k_2 V_c(s) - \frac{V_{out}(s)}{Z(s)}$$

$I_{out}(s)$

Rearranging the Expression

- Further to a few manipulations, you should find

$$H(s)=\frac{V_{out}(s)}{D(s)}=\frac{R_1k_3+R_1k_1k_5}{k_5-R_1+R_1k_4}\frac{\left(1-sL\frac{k_1-k_2k_3+k_1k_6}{k_3+k_1k_5}\right)}{1+s\left(\frac{R_1k_5C_1+L(R_1k_2k_4-1-k_6-R_1k_2)}{k_5-R_1+R_1k_4}\right)+s^2C_1L\frac{R_1+R_1k_6}{R_1-k_5-R_1k_4}}$$

- This is a second-order system affected by a RHP zero

$$H(s)=G_0\frac{1-\frac{s}{s_{z_1}}}{1+\frac{s}{\omega_0Q}+\left(\frac{s}{\omega_0}\right)^2}$$

$$G_0=\frac{R_1k_3+R_1k_1k_5}{k_5-R_1+R_1k_4}$$

$$\omega_{z_1}=\frac{1}{L\left(\frac{k_1-k_2k_3+k_1k_6}{k_3+k_1k_5}\right)}$$

$$Q=\frac{k_5/R_1-1+k_4}{\omega_0\left(L\left(\frac{k_6}{R_1}+k_2-k_2k_4+\frac{1}{R_1}\right)-C_1k_5\right)}$$

$$\omega_0=\frac{1}{\sqrt{LC_1}}\sqrt{\frac{1-\frac{k_5}{R_1}-k_4}{1+k_6}}$$

A Low-Q System

- A second-order system features two poles
- These poles are the characteristic equation roots

$$D(s) = 1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2 \xrightarrow{D(s)=0} s_{p_1}, s_{p_2} = \frac{\omega_0}{Q}\frac{1 \pm \sqrt{1-4Q^2}}{2}$$

- When Q is low, use MacLaurin series to simplify

$$x \ll 1 \longrightarrow (1+x)^n \approx 1 + nx \longrightarrow \sqrt{1+x} \approx 1 + \frac{1}{2}x$$

$$s_{p_1} = \frac{\omega_0}{Q}\frac{1+\sqrt{1-4Q^2}}{2} \approx -\frac{\omega_0}{Q}\left(Q^2 - 1\right) \approx \frac{\omega_0}{Q}$$

$$s_{p_2} = \frac{\omega_0}{Q}\frac{1-\sqrt{1-4Q^2}}{2} \approx \frac{\omega_0}{Q}\frac{1-\left(1-2Q^2\right)}{2} \approx Q\omega_0$$

$$D(s) \approx \left(1 + \frac{s}{s_{p_1}}\right)\left(1 + \frac{s}{s_{p_2}}\right)$$

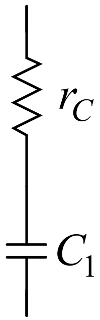
- A 2nd-order low-Q system is equivalent to 2 cascaded poles

A Simpler Expression

- We can thus rewrite a simpler transfer function

$$H(s) \approx G_0 \frac{\left(1-\frac{s}{s_{z_1}}\right)\left(1+\frac{s}{s_{z_2}}\right)}{\left(1+\frac{s}{s_{p_1}}\right)\left(1+\frac{s}{s_{p_2}}\right)}$$

← Adding the ESR



$$\frac{s r_C C_1 + 1}{s C_1} = 0$$

- Further to rearranging, simplifying and having fun…

$$G_0 \approx \frac{2}{2M-1}\sqrt{\frac{M(M-1)R_1}{2F_{sw}L}} \qquad \omega_{p_1} \approx \frac{2M-1}{M}\frac{1}{R_1 C_1} \qquad \omega_{p_2} \approx 2F_{sw}\left(\frac{1-1/M}{D}\right)^2 \geq 2F_{sw}$$

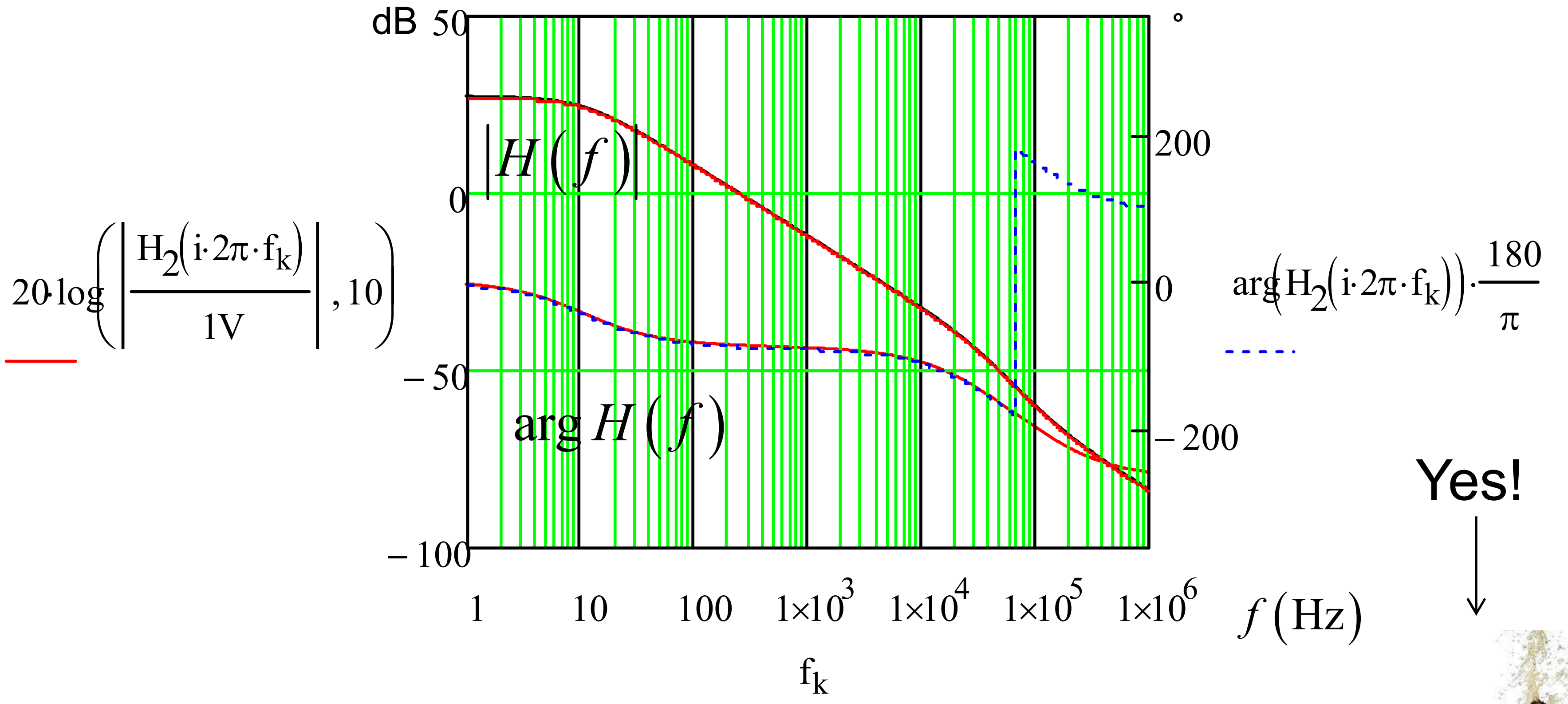
$$\omega_{z_1} \approx \frac{R_2}{M^2 L} \geq 2F_{sw} \qquad \omega_{z_2} \approx \frac{1}{r_C C_1}$$

- The DCM model is of the same order as the CCM model
- The difference is in the damping

V. Vorpérian,"Analytical Methods in Power Electronics", In-house class, Toulouse, 2004

Full Mathcad® Expression versus SPICE

- The curves perfectly superimpose, calculations are correct!

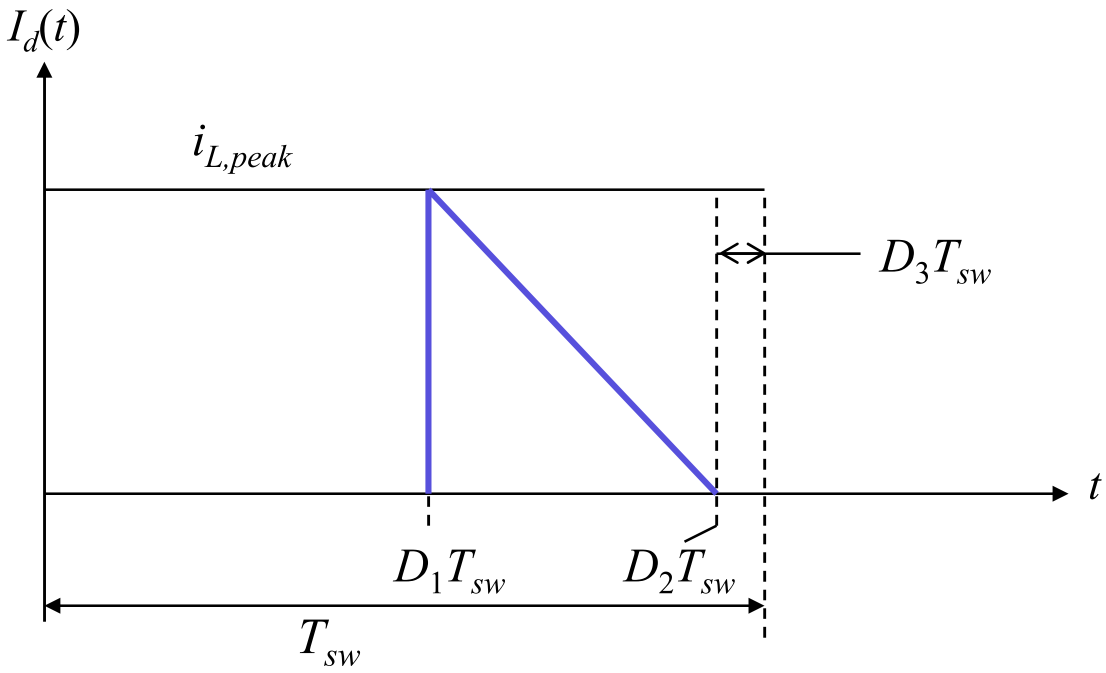


Yes!

$$H_2(s) := \frac{(R_1 \cdot k_3 + R_1 \cdot k_1 \cdot k_5)}{(k_5 - R_1 + R_1 \cdot k_4)} \cdot \frac{\left[1 - s \cdot L_1 \cdot \left(\frac{k_1 - k_2 \cdot k_3 + k_1 \cdot k_6}{k_3 + k_1 \cdot k_5}\right)\right]}{1 + s \cdot \left[\frac{R_1 \cdot k_5 \cdot C_1 + L_1 \cdot (R_1 \cdot k_2 \cdot k_4 - 1 - k_6 - R_1 \cdot k_2)}{k_5 - R_1 + R_1 \cdot k_4}\right] + s^2 \cdot C_1 \cdot L_1 \cdot \frac{(R_1 + R_1 \cdot k_6)}{(-k_5 + R_1 - R_1 \cdot k_4)}}$$

Why a RHP Zero in DCM?

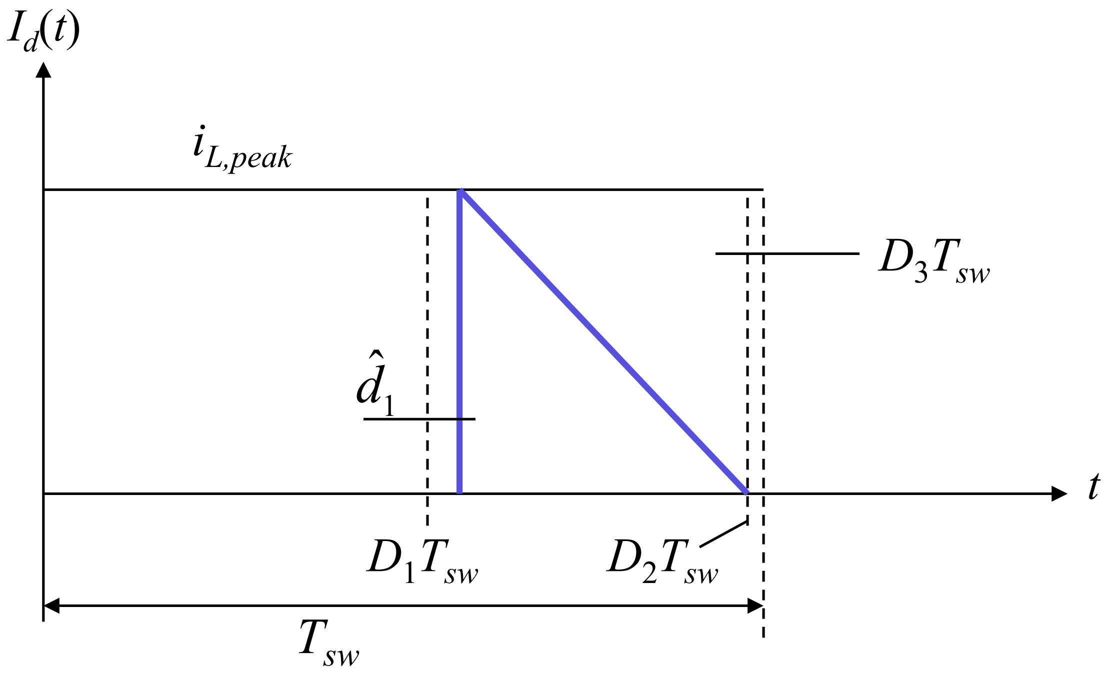
❑ The RHP Zero is present in CCM and is still there in DCM!



❑ When D_1 increases, $[D_1, D_2]$ stays constant but D_3 shrinks

Why a RHP Zero in DCM?

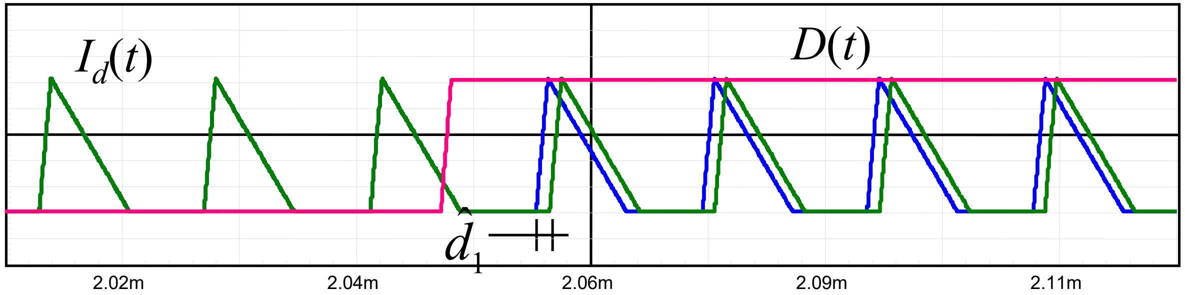
- The triangle is simply shifted to the right by $\hat{d}_1$

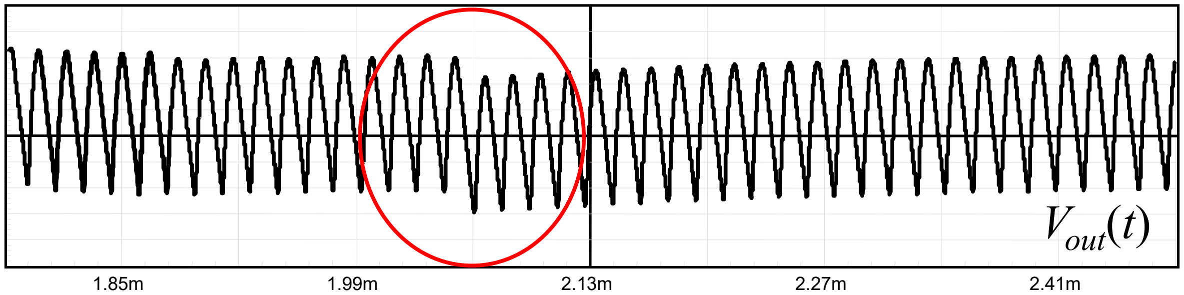


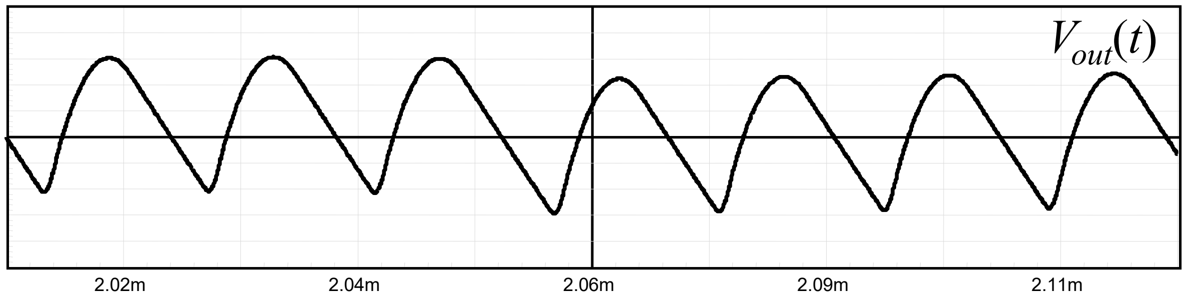
- The capacitor refueling time is delayed and a drop occurs

The Refueling Time is Shifted

- If D increases, the diode current is delayed by $\hat{d}_1$







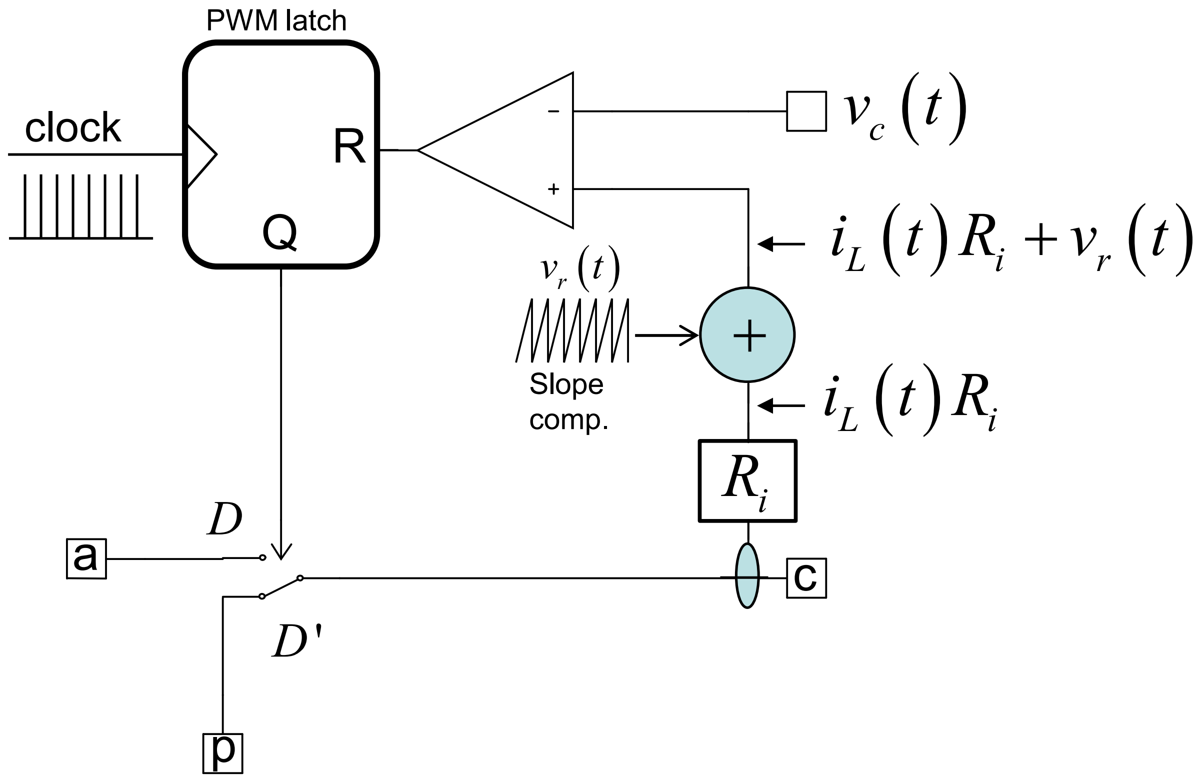
Simulation, no peak increase

Course Agenda

- ❑ Linear and Non-Linear Functions
- ❑ What is a Small-Signal Model?
- ❑ Fast Analytical Techniques at Work
- ❑ From a Switched to Linearized Model
- ❑ The CCM VM Small-Signal PWM Switch Model
- ❑ The DCM VM Small-Signal PWM Switch Model
- ❑ **Peak Current Mode Control in Large Signal**
- ❑ The CCM CM Small-Signal PWM Switch Model
- ❑ The DCM CM Small-Signal PWM Switch Model
- ❑ The PWM Switch in Boundary Mode

Peak Current Mode Control

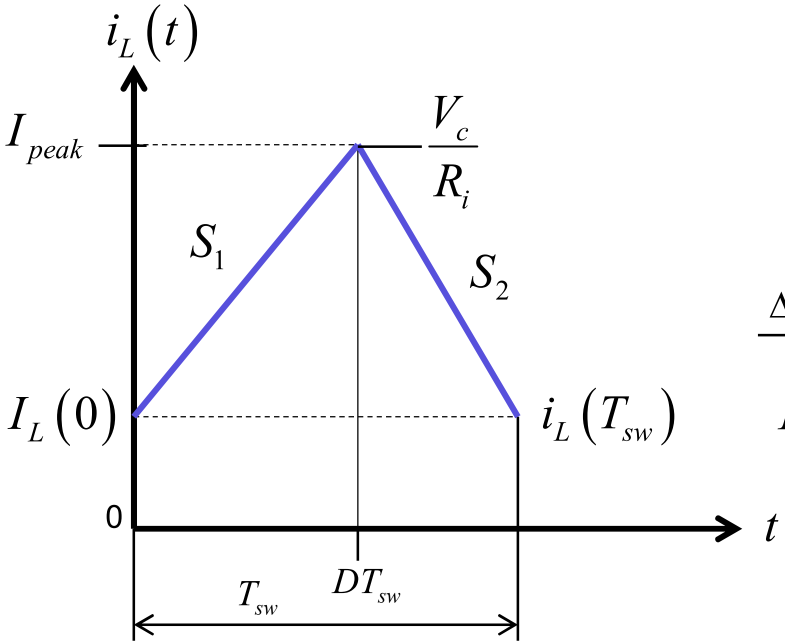
- ❑ In voltage-mode, the loop controls the duty ratio
- ❑ In current-mode, the inductor peak current is controlled

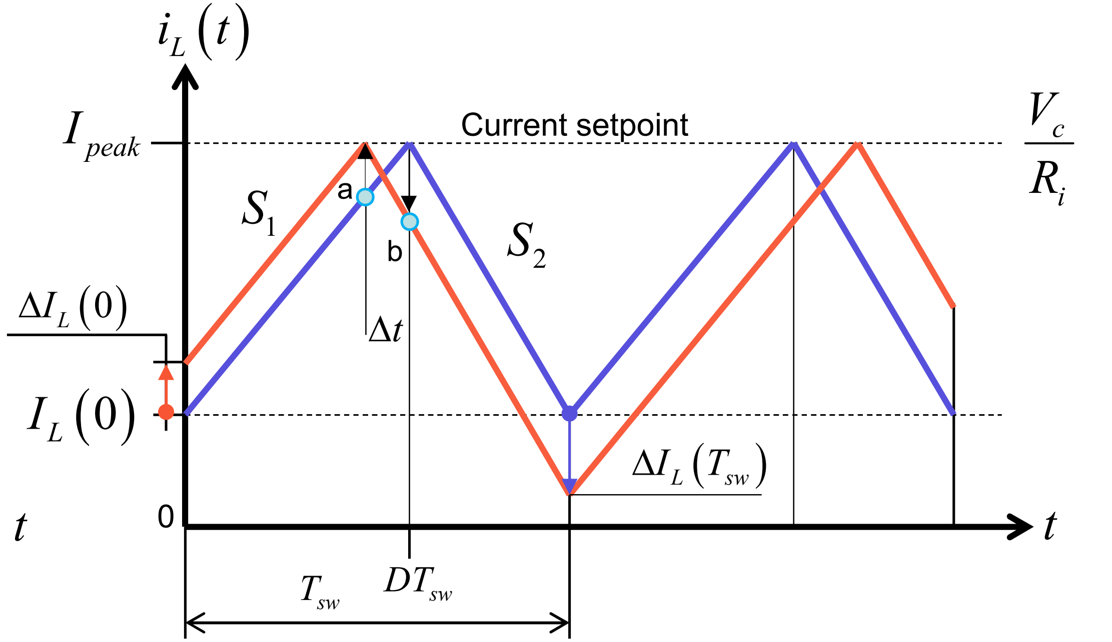


- ❑ An artificial ramp is added for stabilization purposes

How Does a Pertubation Propagate?

- ❑ The perturbation is carried cycle by cycle in CCM





$$I_L(T_{sw}) = I_L(0) + \underbrace{\boxed{S_1 DT_{sw} - S_2 D'T_{sw}}}_{=0}$$

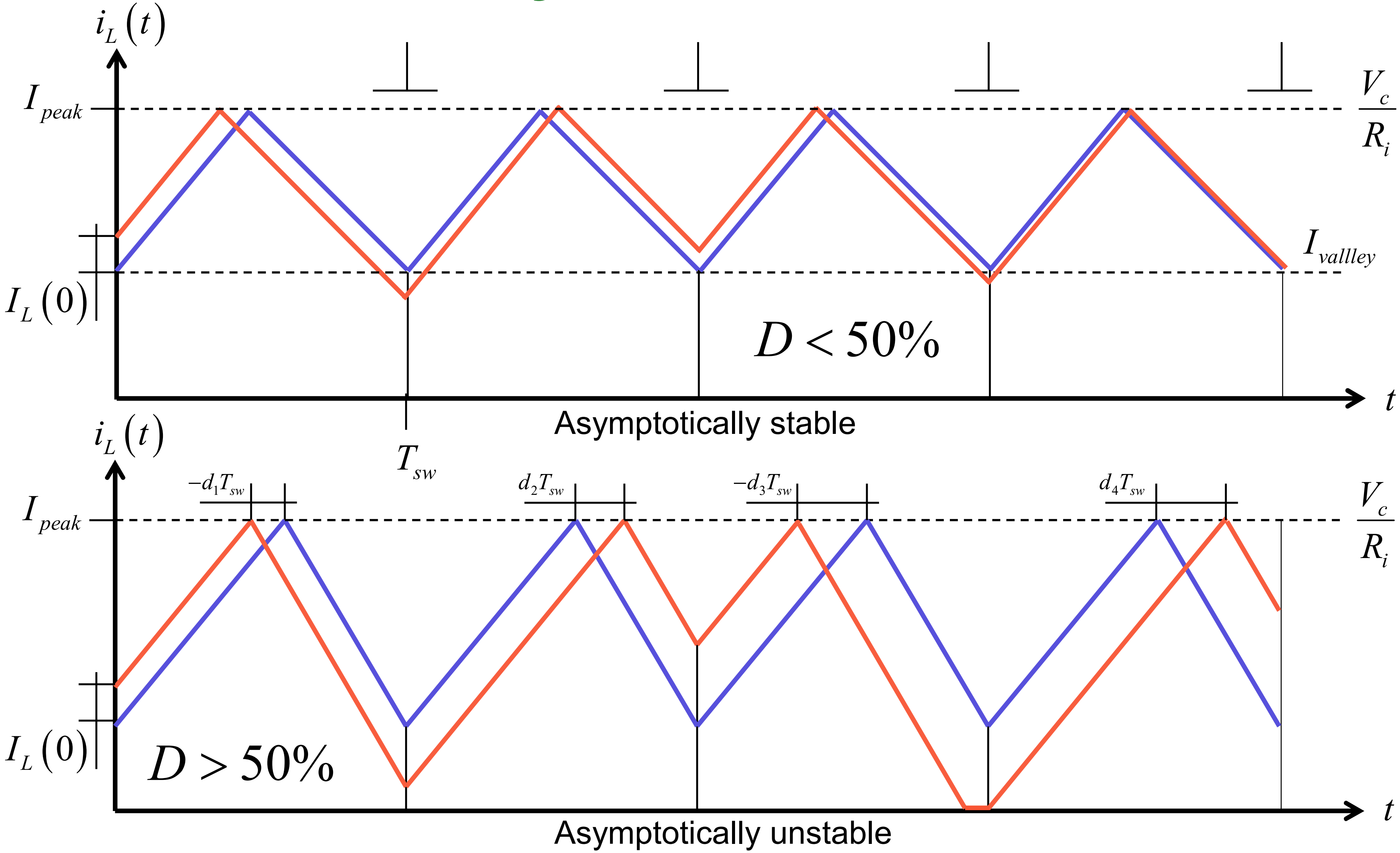
$$S_1 DT_{sw} = S_2 D'T_{sw} \longrightarrow \frac{S_2}{S_1} = \frac{D}{D'}$$

$$I_{peak} = a + S_1 \Delta t \qquad b = I_{peak} - S_2 \Delta t$$

$$\frac{\overbrace{I_{peak} - a}^{\Delta I_L(0)}}{S_1} = \frac{\overbrace{I_{peak} - b}^{\Delta I_L(T_{sw})}}{S_2}$$

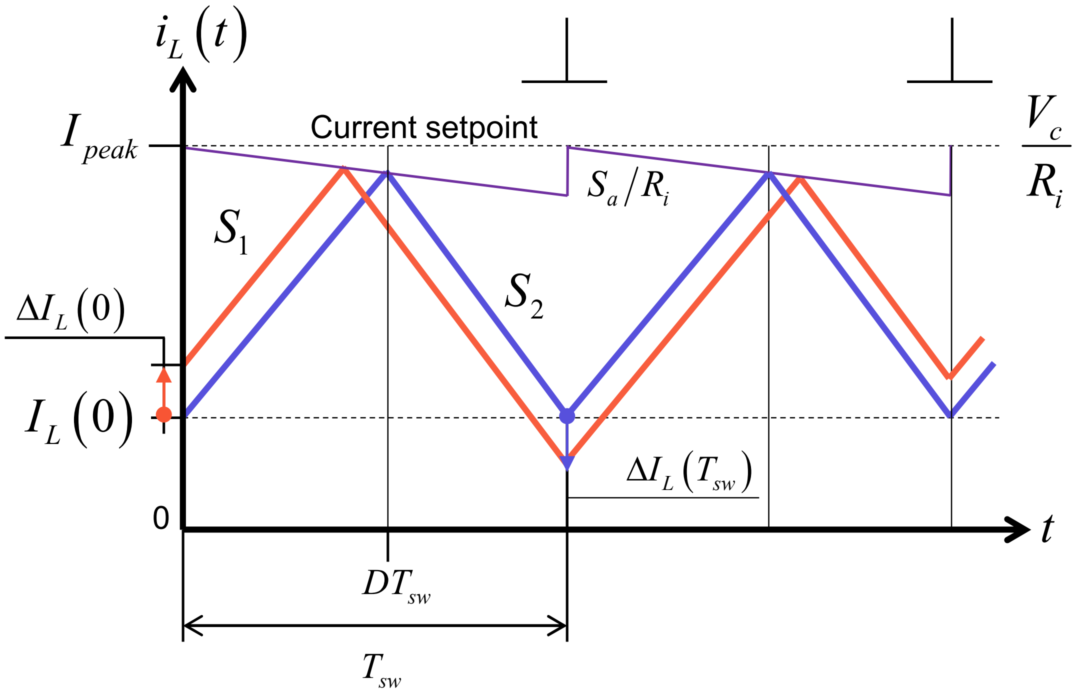
generalized $\Longrightarrow$ $$\Delta I_L(nT_{sw}) = \Delta I_L(0)\left(-\frac{D}{D'}\right)^n$$

Instability in CCM for D > 50%



Add an Artificial Ramp

- ❑ An external ramp helps to stabilize the current



$$\Delta I_L(nT_{sw}) = \Delta I_L(0)\left[-\frac{1-\frac{S_a}{S_2}}{\frac{D'}{D}+\frac{S_a}{S_2}}\right]^n$$

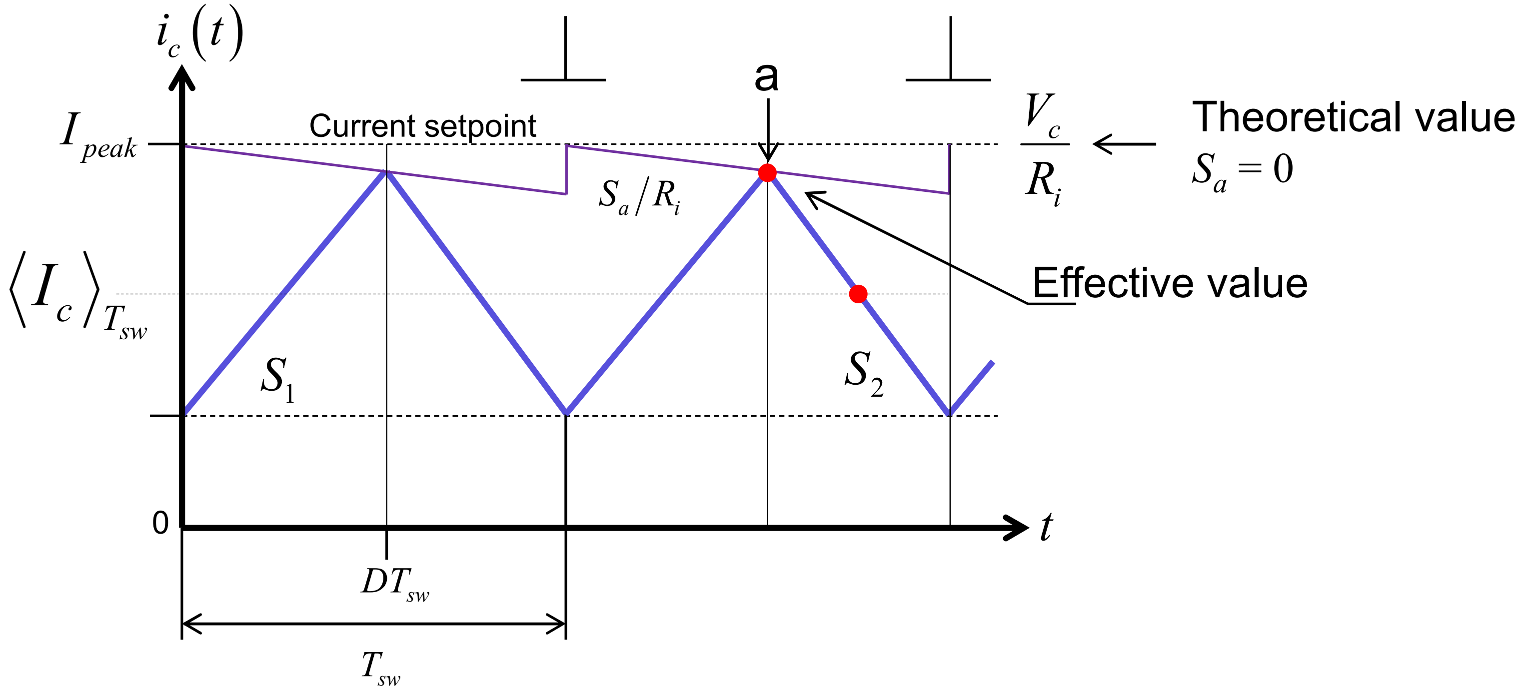
0 for 100% duty ratio

$$\left|\frac{1-\frac{S_a}{S_2}}{0+\frac{S_a}{S_2}}\right| < 1 \qquad S_a > 50\% S_2$$

- ❑ Considering 100% duty ratio overcompensates the design

What is the Control Law in CCM?

❑ Draw the primary current with a stabilization ramp S_a



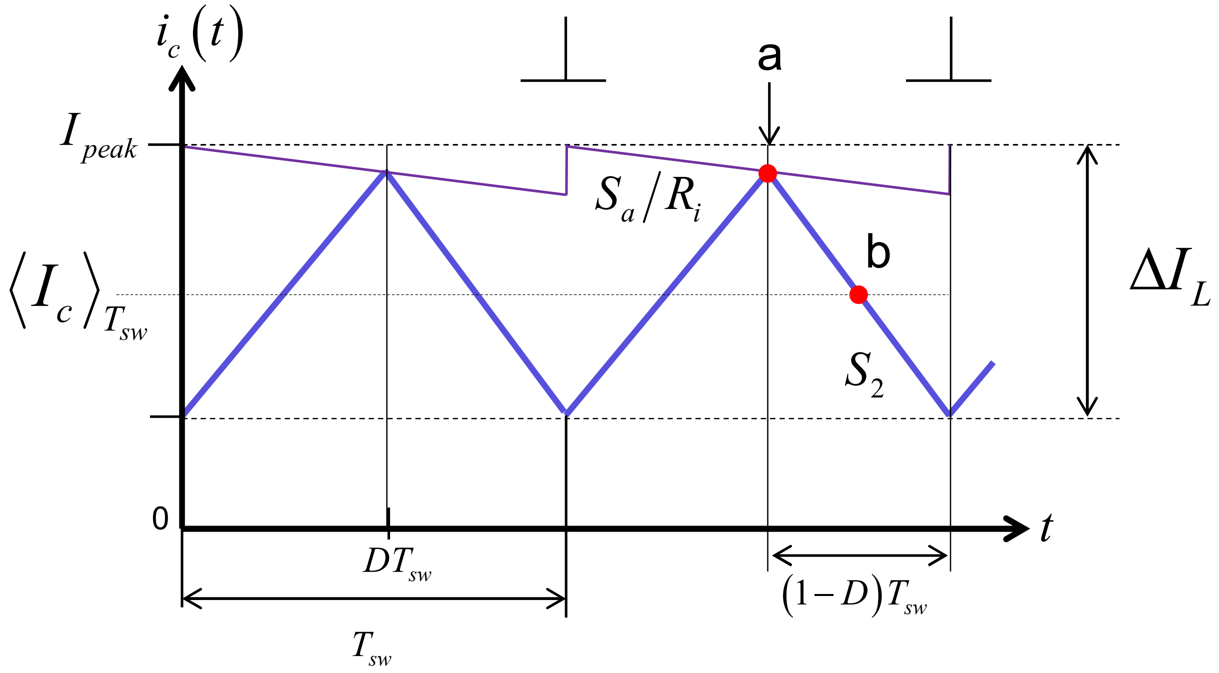
❑ The current at point **a** is defined by:

$$I_{c(a)} = \frac{V_c}{R_i} - \frac{S_a D T_{sw}}{R_i}$$

ON Semiconductor®

We Want the Average Current Definition

- The value I_c is the inductor current at half the ripple

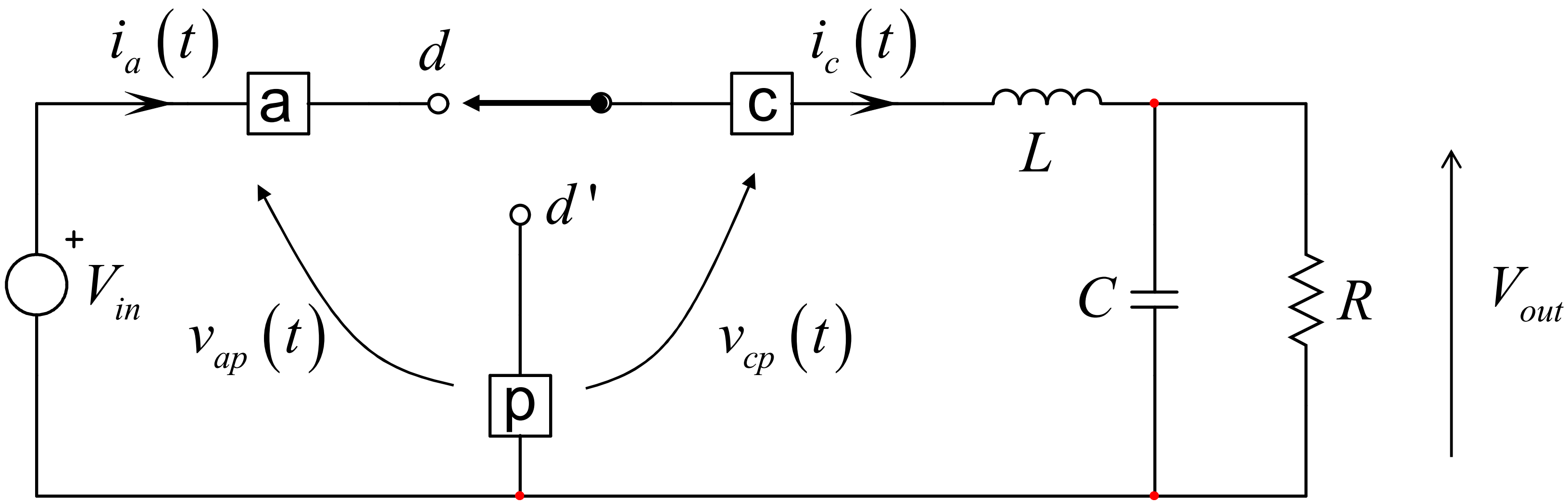


- Express current at point **b** and substitute the **a** definition

$$I_{c(b)} = \langle I_c \rangle_{T_{sw}} = I_{c(a)} - \frac{\Delta I_L}{2} = I_{c(a)} - \frac{S_2 D' T_{sw}}{2} \longrightarrow \langle I_c \rangle_{T_{sw}} = \frac{V_c}{R_i} - \frac{S_a}{R_i} D T_{sw} - \frac{S_2 D' T_{sw}}{2}$$

Define the Converter off-Slope

- Use a buck configuration to see voltages at play



- The downslope depends on the output voltage V_{out}:

$$S_2 = -\frac{V_{out}}{L}$$

- The inductor average voltage is 0 at steady-state

$$V_{cp} = V_{out} \longrightarrow S_2 = -\frac{V_{cp}}{L}$$

A Current Mode Generator

- Update the previous equation to obtain final definition

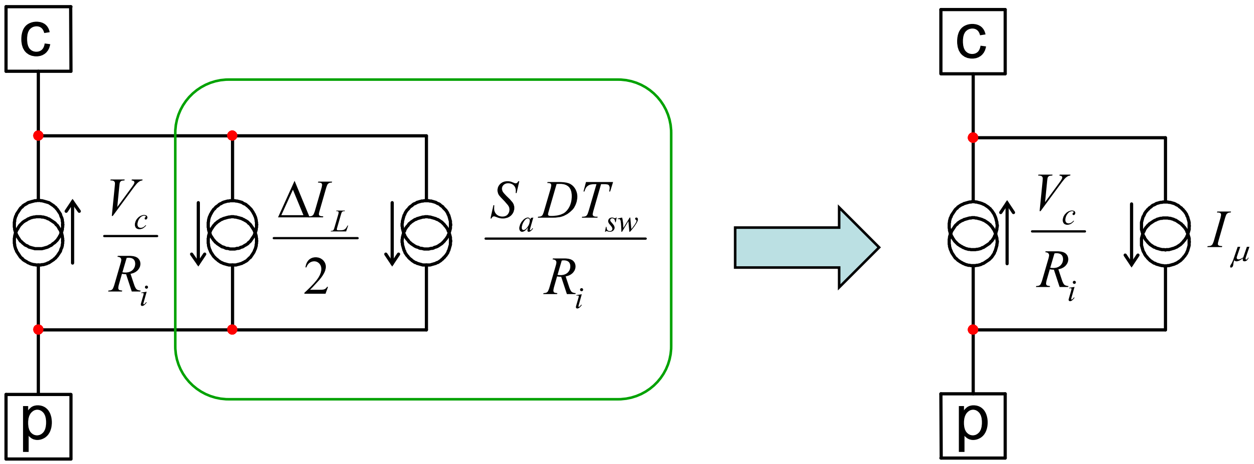
$$I_c = \frac{V_c}{R_i} - V_{cp}(1-D)\frac{T_{sw}}{2L} - \frac{S_a}{R_i}DT_{sw}$$

Peak current setpoint | Half inductor ripple | Compensation ramp

Group 2nd and 3rd terms →

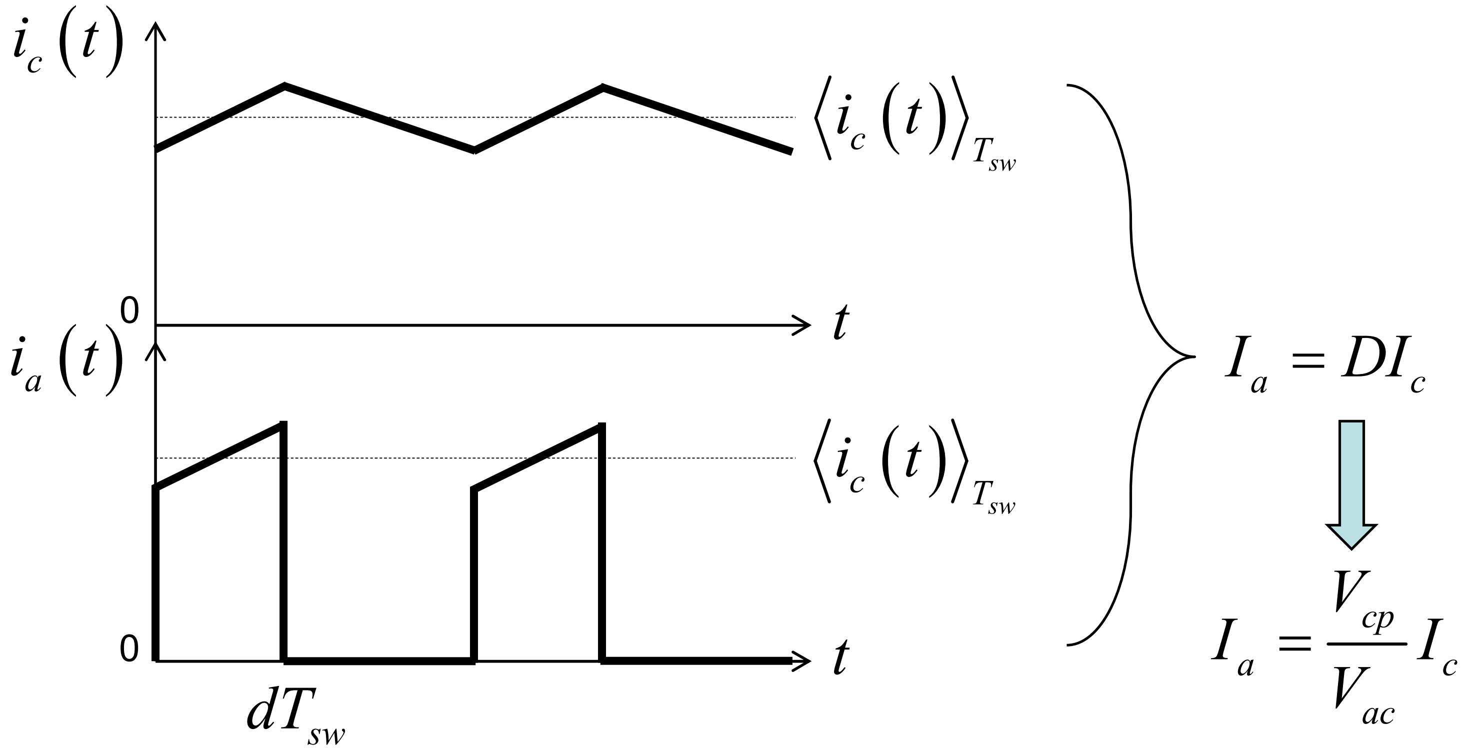
$$I_\mu = V_{cp}(1-D)\frac{T_{sw}}{2L} + \frac{S_a}{R_i}DT_{sw}$$

- Inductor ripple and compensation ramp alter peak value



CM or VM Lead to Similar Input Currents

- Average the current waveforms across the PWM switch

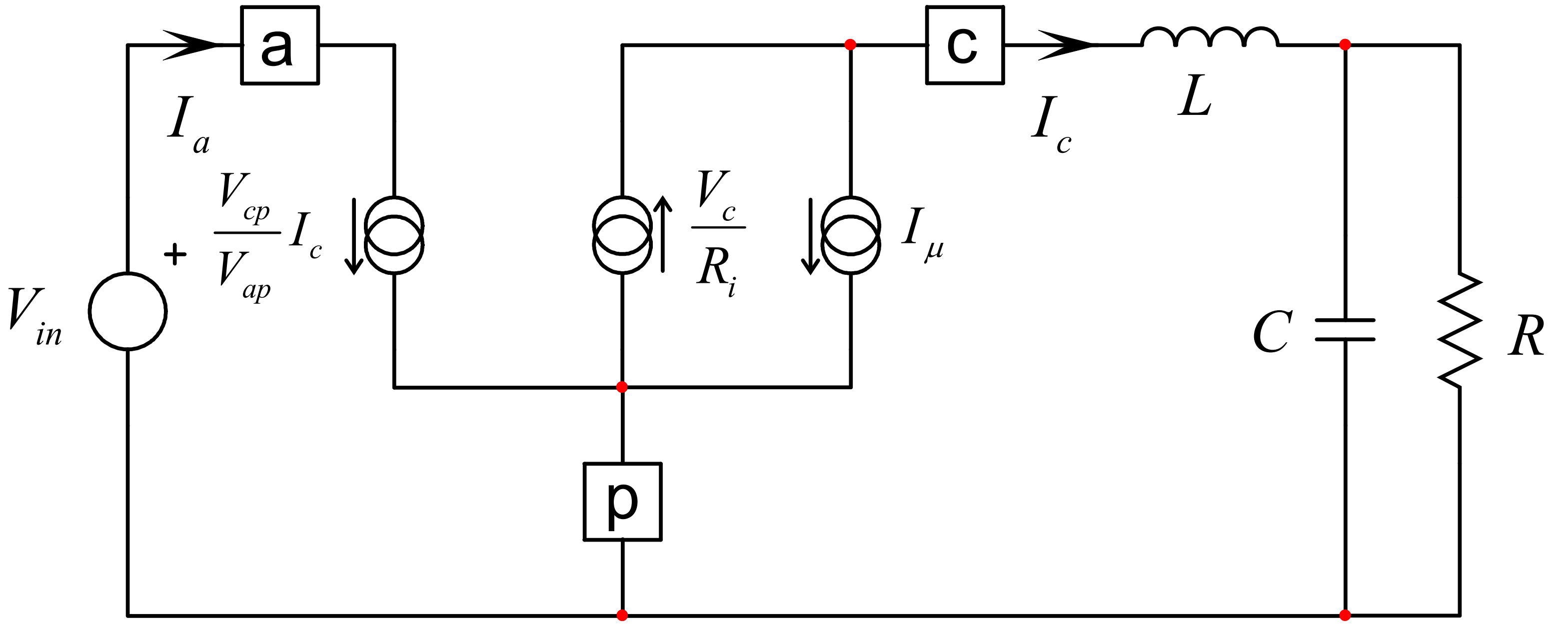


$$\langle i_a(t)\rangle_{T_{sw}} = I_a = \frac{1}{T_{sw}}\int_0^{dT_{sw}} i_a(t)\,dt = D\langle i_c(t)\rangle_{T_{sw}} = DI_c \qquad D = \frac{V_{cp}}{V_{ap}}$$

CCM

The PWM Switch Model in Current Mode

❑ The final model associates three current sources



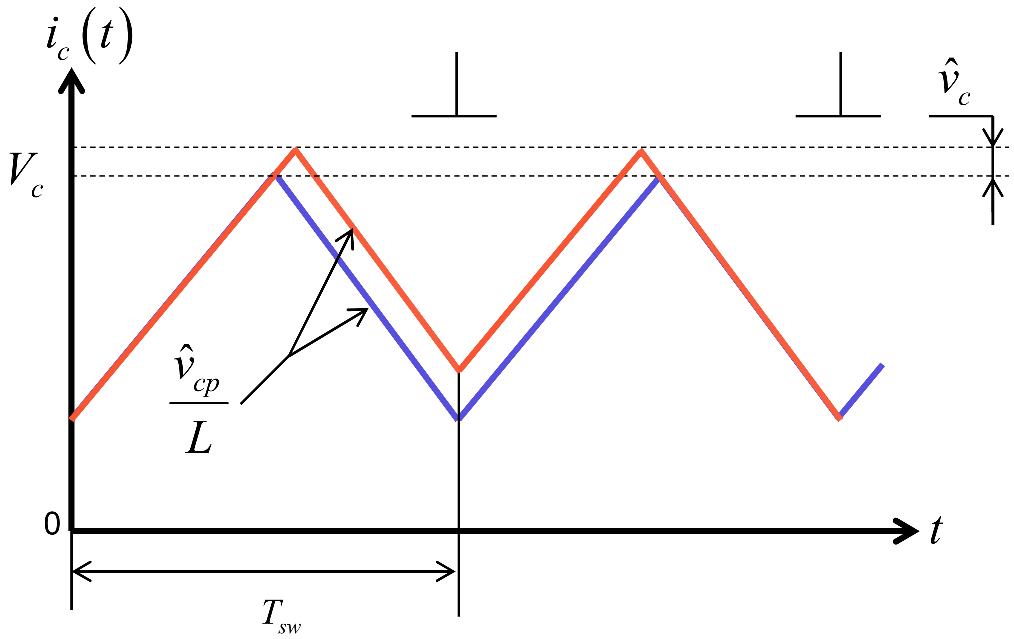
❑ This is the large-signal current-mode PWM switch model

Can you think of something simpler?!

V. Vorpérian,"Analysis of Current-Controlled PWM Converters using the Model of the PWM Switch", PCIM Conference ,1990

How to Model Subharmonic Instabilities?

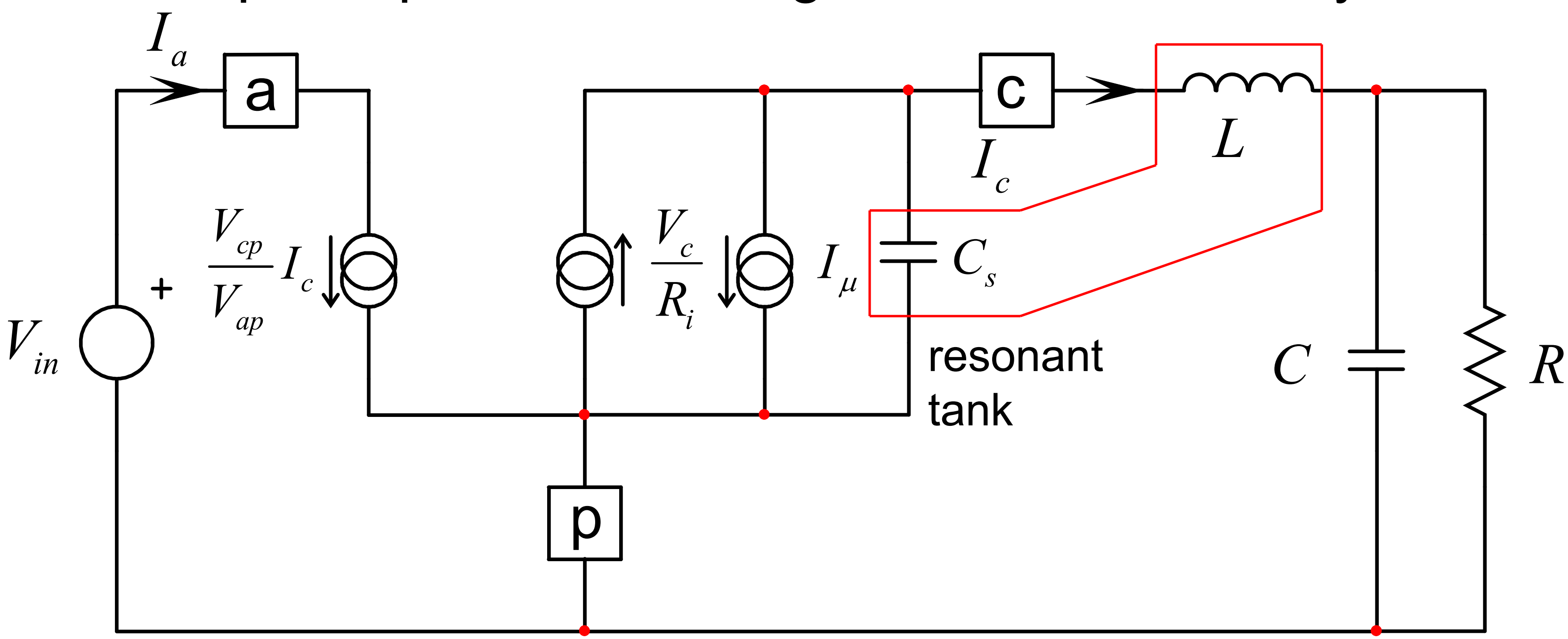
- ❑ The model, as it is, cannot predict instabilities
- ❑ Let's observe a small-signal perturbation in v_c



- ❑ The off-slope does not change as $\hat{v}_{cp}$ keeps constant
- ❑ This "memory" effect is modeled with a capacitor

Final Model Includes Subharmonic Effects

- A simple capacitor is enough to mimic instability

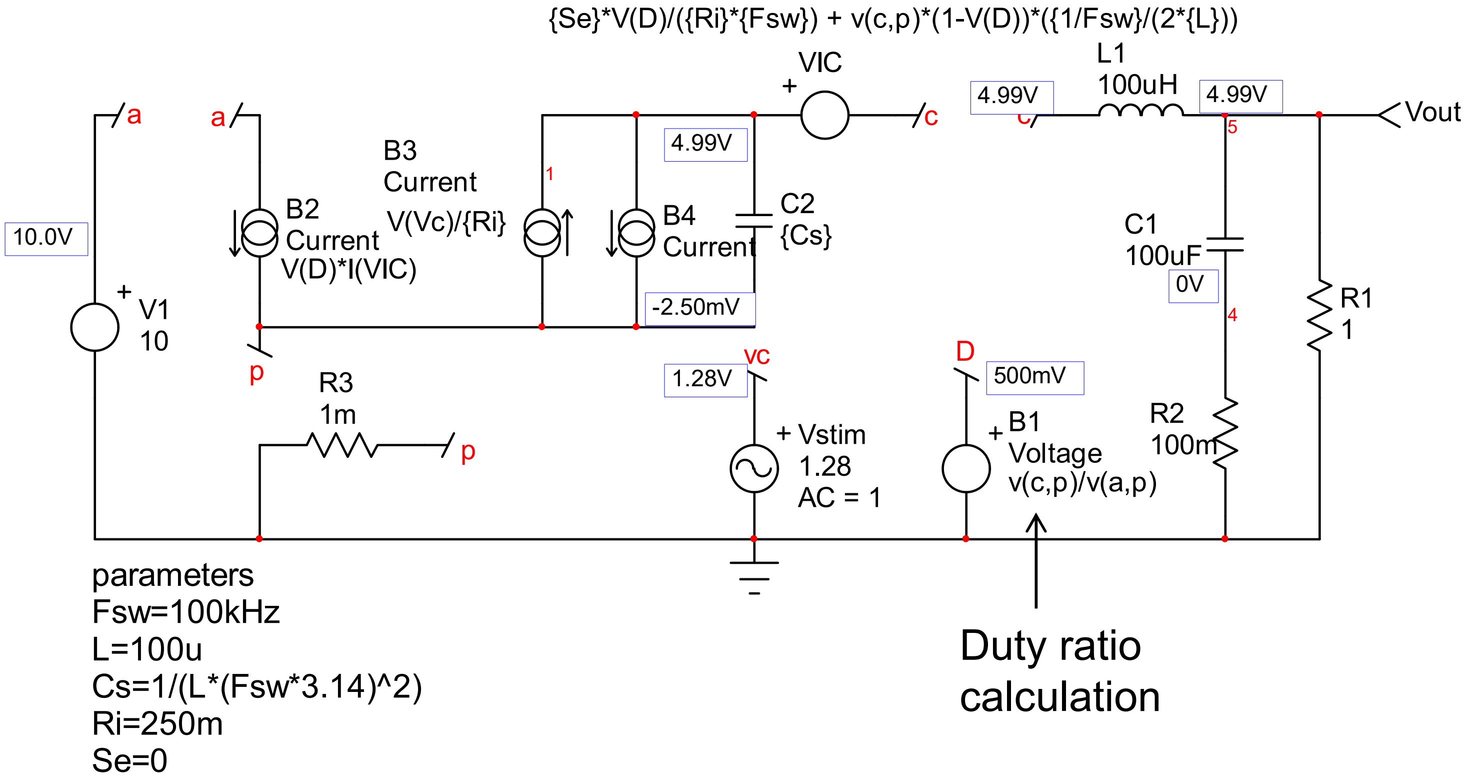


- As the instability is placed at half the switching frequency:

$$\frac{F_{sw}}{2} = \frac{1}{2\pi\sqrt{LC_s}} \quad \Rightarrow \quad C_s = \frac{1}{L\left(F_{sw}\pi\right)^2}$$

The Large-Signal Model at Work

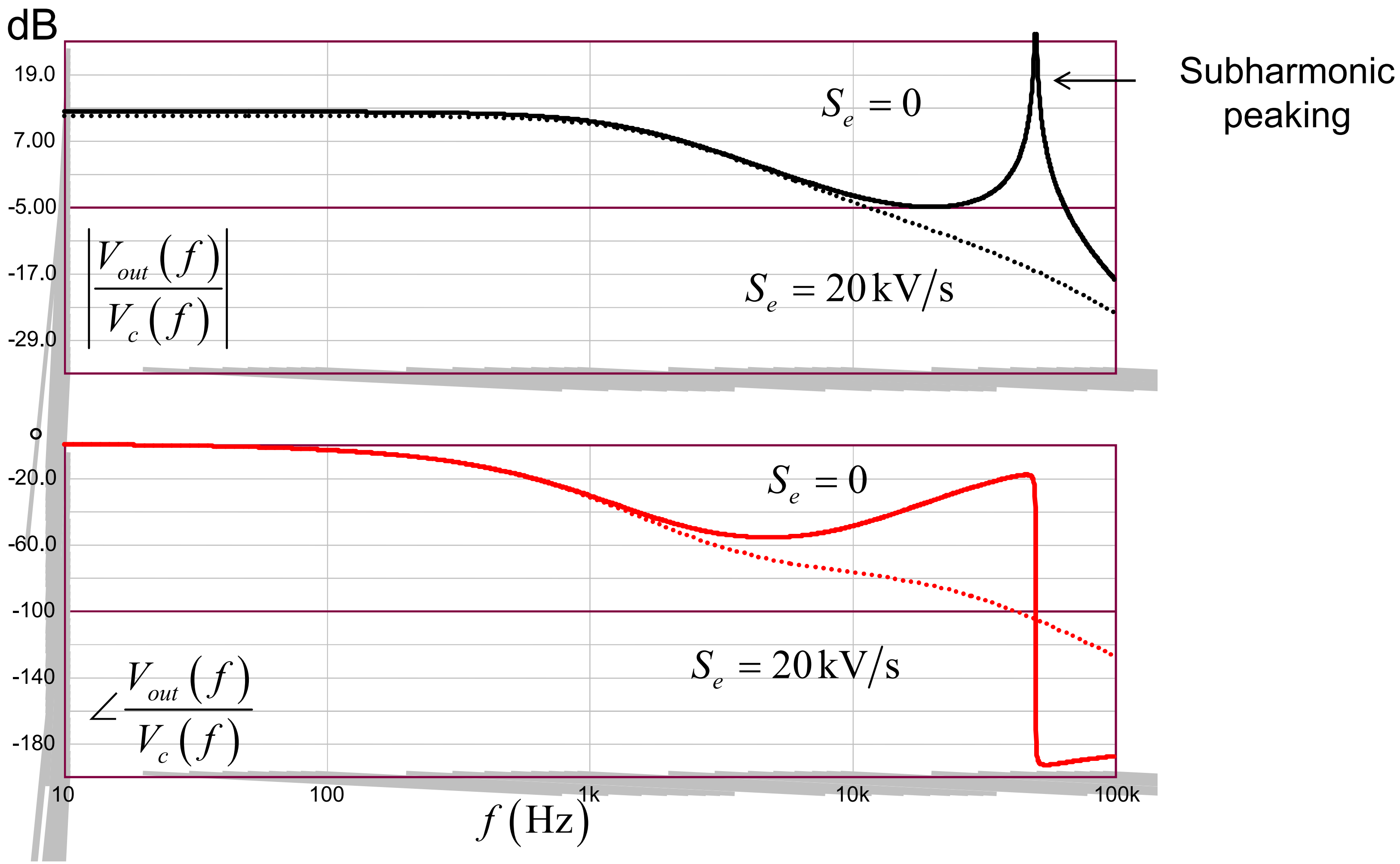
- ❑ A current-mode 5-V/5-A buck converter as an example



- ❑ Ramp compensation is adjusted to see its effects

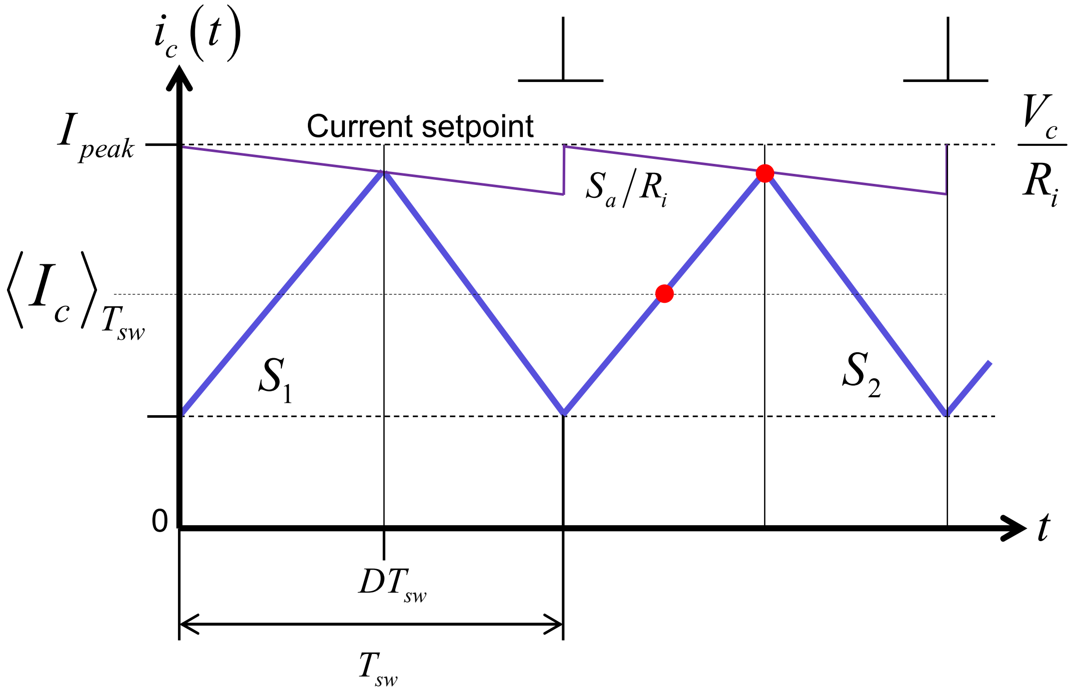
Instabilities Show up as Expected

- ❑ Peaking is tamed by increasing slope compensation



Another Way of Modeling

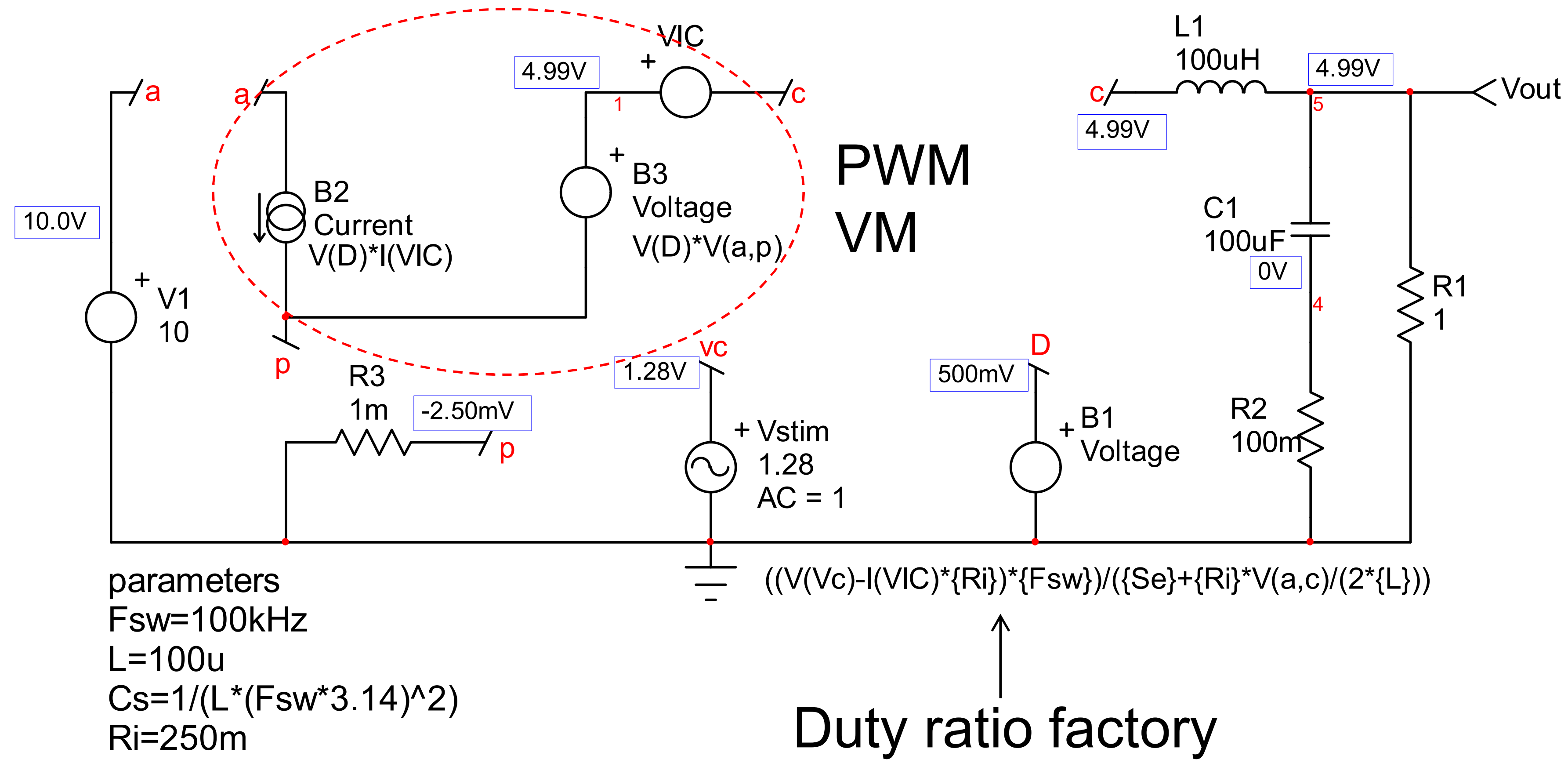
- We can build a duty ratio factory and use the VM model



$$I_c = \frac{V_c}{R_i} - \frac{S_a}{R_i} DT_{sw} - \frac{V_{ac}}{2L} DT_{sw} \quad \Rightarrow \quad D = \frac{F_{sw}\left(V_c - R_i I_c\right)}{S_a + \frac{R_i V_{ac}}{2L}}$$

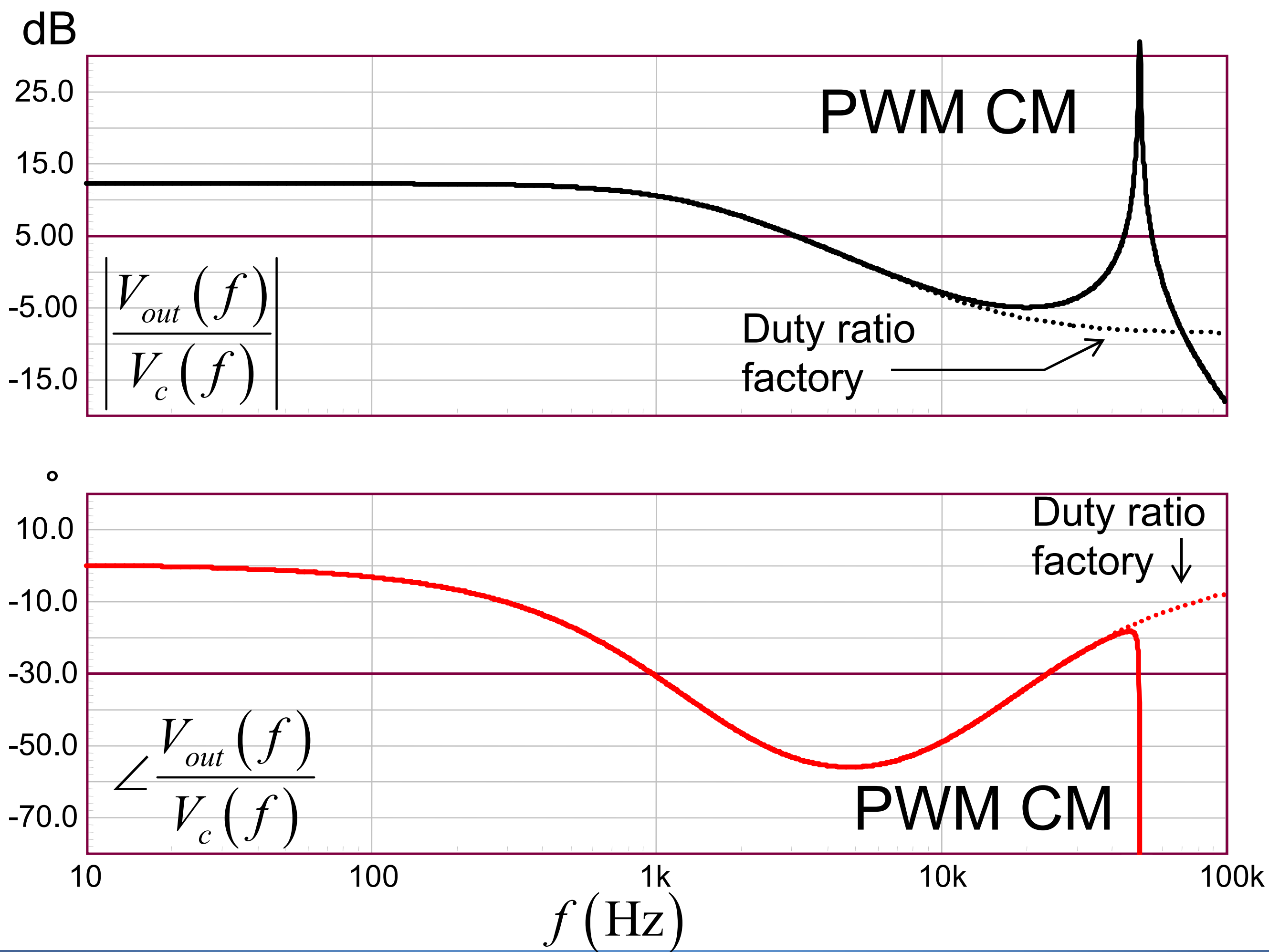
Build a Duty Ratio Factory

- An in-line equation builds D from the control voltage V_c



No Sub-Harmonic Oscillations Prediction

- Low-frequency gains are identical but no peaking at $F_{sw}/2$



$S_e = 0$

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Small-Signal Modeling of PWM CM

- There are two equations to linearize:

$$I_c = \frac{V_c}{R_i} - V_{cp}\left(1 - \frac{V_{cp}}{V_{ap}}\right)\frac{T_{sw}}{2L} - \frac{S_a}{R_i}\frac{V_{cp}}{V_{ap}}T_{sw} \longrightarrow \text{3 variables}$$

$$I_a = \frac{V_{cp}}{V_{ap}} I_c = \frac{V_{cp}}{V_{ap}}\left(\frac{V_c}{R_i} - V_{cp}\left(1 - \frac{V_{cp}}{V_{ap}}\right)\frac{T_{sw}}{2L} - \frac{S_a}{R_i}\frac{V_{cp}}{V_{ap}}T_{sw}\right) \longrightarrow \text{3 variables}$$

- Apply partial differentiation:

$$\hat{i}_c = \frac{\partial I_c\left(V_c, V_{ap}, V_{cp}\right)}{\partial V_c}\hat{v}_c + \frac{\partial I_c\left(V_c, V_{ap}, V_{cp}\right)}{\partial V_{ap}}\hat{v}_{ap} + \frac{\partial I_c\left(V_c, V_{ap}, V_{cp}\right)}{\partial V_{cp}}\hat{v}_{cp}$$

$$\hat{i}_a = \frac{\partial I_a\left(V_c, V_{ap}, V_{cp}\right)}{\partial V_c}\hat{v}_c + \frac{\partial I_a\left(V_c, V_{ap}, V_{cp}\right)}{\partial V_{ap}}\hat{v}_{ap} + \frac{\partial I_a\left(V_c, V_{ap}, V_{cp}\right)}{\partial V_{cp}}\hat{v}_{cp}$$

Small-Signal Modeling of PWM CM

- Rearrange the coefficients and rewrite equations:

$$\hat{i}_c = k_o \hat{v}_c + g_f \hat{v}_{ap} + g_o \hat{v}_{cp} \qquad \hat{i}_a = k_i \hat{v}_c + g_i \hat{v}_{ap} + g_r \hat{v}_{cp}$$

Where:

$$k_o = \frac{1}{R_i} \qquad g_f = \frac{S_a T_{sw} D}{R_i V_{ap}} - \frac{T_{sw} D^2}{2L} \qquad g_o = \frac{T_{sw}(D-1)}{2L} + \frac{T_{sw} D}{2L} - \frac{S_a T_{sw} D}{R_i V_{cp}}$$

$$k_i = \frac{D}{R_i} \qquad g_i = D\left(g_f - \frac{I_c}{V_{ap}}\right) \qquad g_r = \frac{I_c}{V_{ap}} + D\left[\frac{T_{sw}(2D-1)}{2L} - \frac{S_a T_{sw}}{R_i V_{ap}}\right]$$

Using Vorpérian notation and $\hat{i}_c = \ldots - g_o \hat{v}_{cp}$

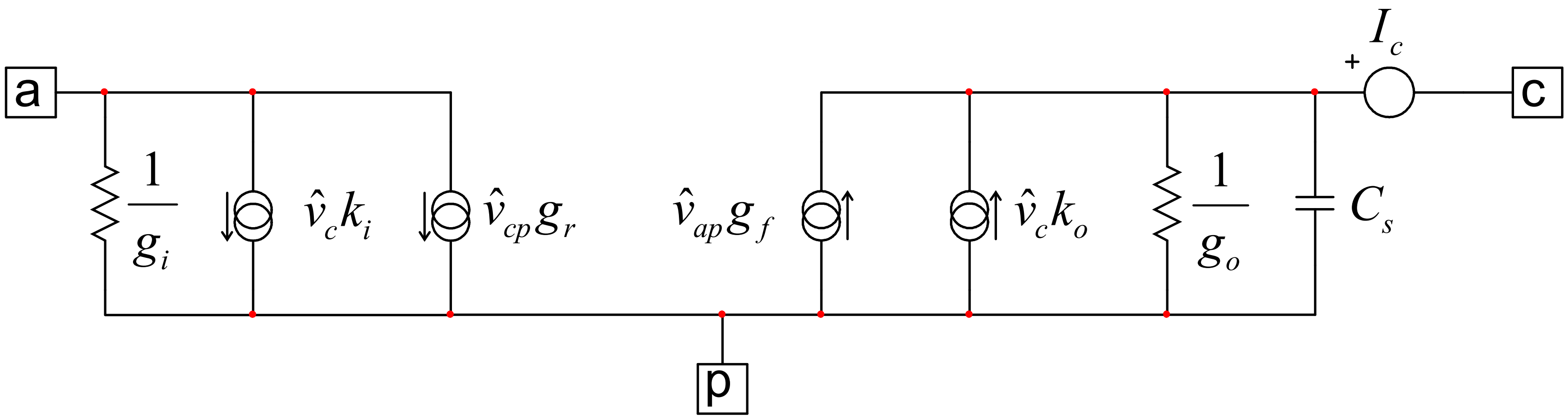
$$k_o = \frac{1}{R_i} \qquad g_f = D g_o - \frac{DD' T_{sw}}{2L} \qquad g_o = \frac{T_{sw}}{L}\left(D' \frac{S_a}{S_n} + \frac{1}{2} - D\right)$$

$$k_i = \frac{D}{R_i} \qquad g_i = D\left(g_f - \frac{I_c}{V_{ap}}\right) \qquad g_r = \frac{I_c}{V_{ap}} - g_o D$$

S_n, on-slope
S_f, off-slope
S_a, external ramp

A Small Signal Model

- The model includes current sources and conductances

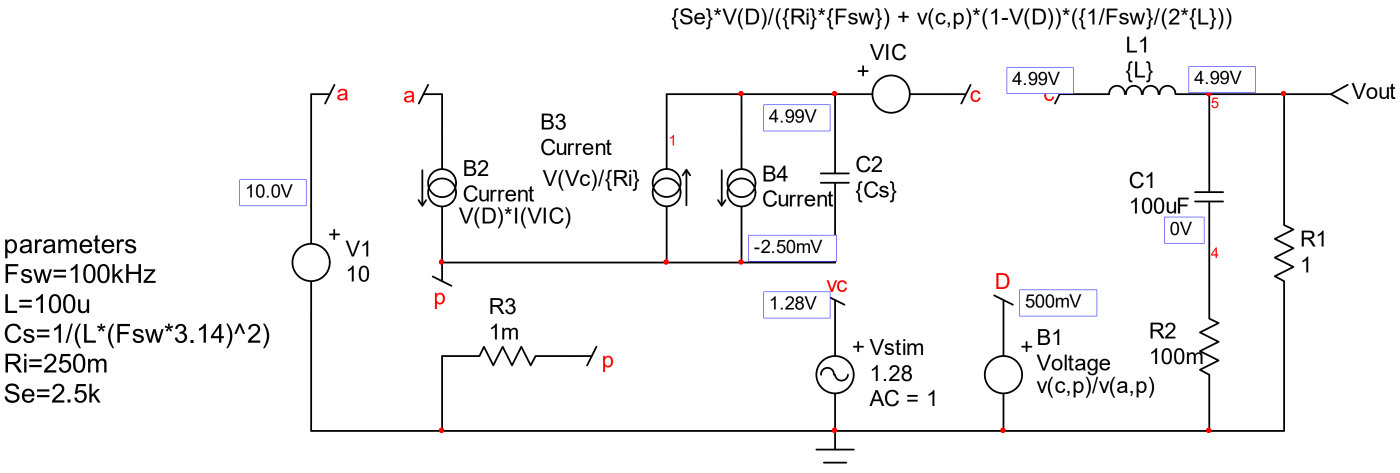


$$k_o = \frac{1}{R_i} \qquad g_f = Dg_o - \frac{DD'T_{sw}}{2L} \qquad g_o = \frac{T_{sw}}{L}\left(D'\frac{S_a}{S_n} + \frac{1}{2} - D\right)$$

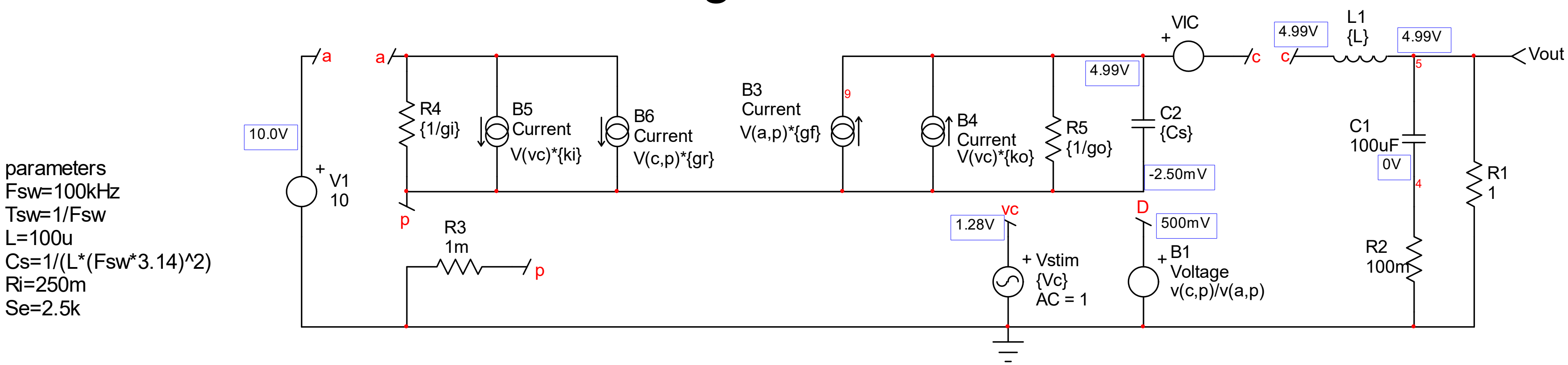
$$k_i = \frac{D}{R_i} \qquad g_i = D\left(g_f - \frac{I_c}{V_{ap}}\right) \qquad g_r = \frac{I_c}{V_{ap}} - g_o D$$

The Model at Work

- Compare ac responses of the linearized large-signal model



- …with that of the small-signal model



Results are Identical

❑ Identical results prove that our small-signal modeling is ok!

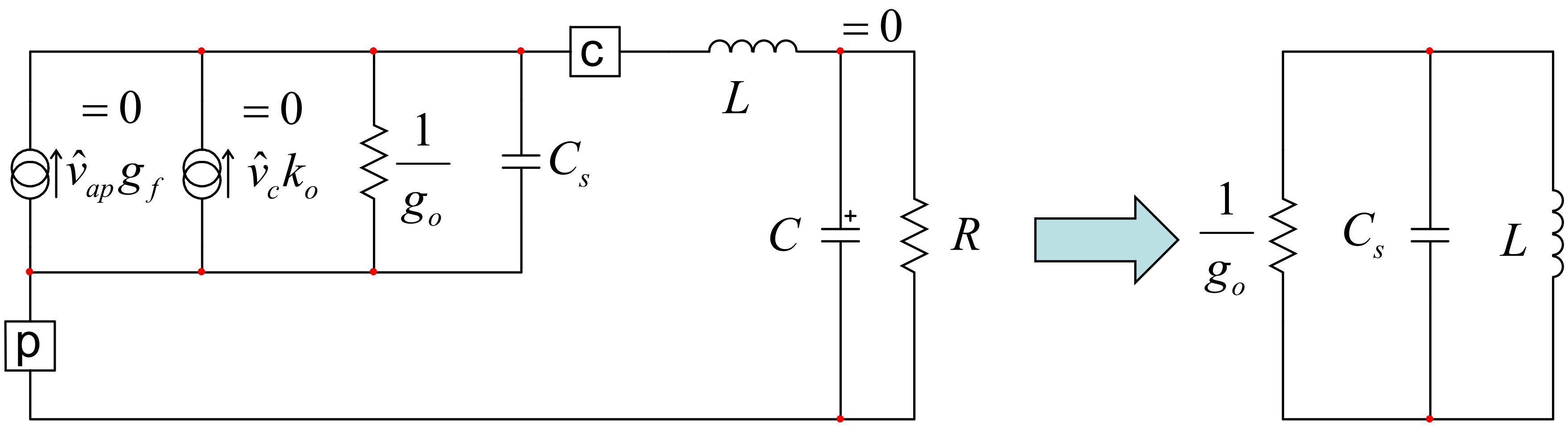
Current Loop Instabilities

- Let's assume input and output voltages are constant

$$V_c = \text{constant} \rightarrow \hat{v}_c = 0$$

$$V_{in}, V_{out} = \text{constant} \rightarrow V_{ap} = \text{constant} \rightarrow \hat{v}_{ap} = 0$$

- The buck implementation simplifies to:

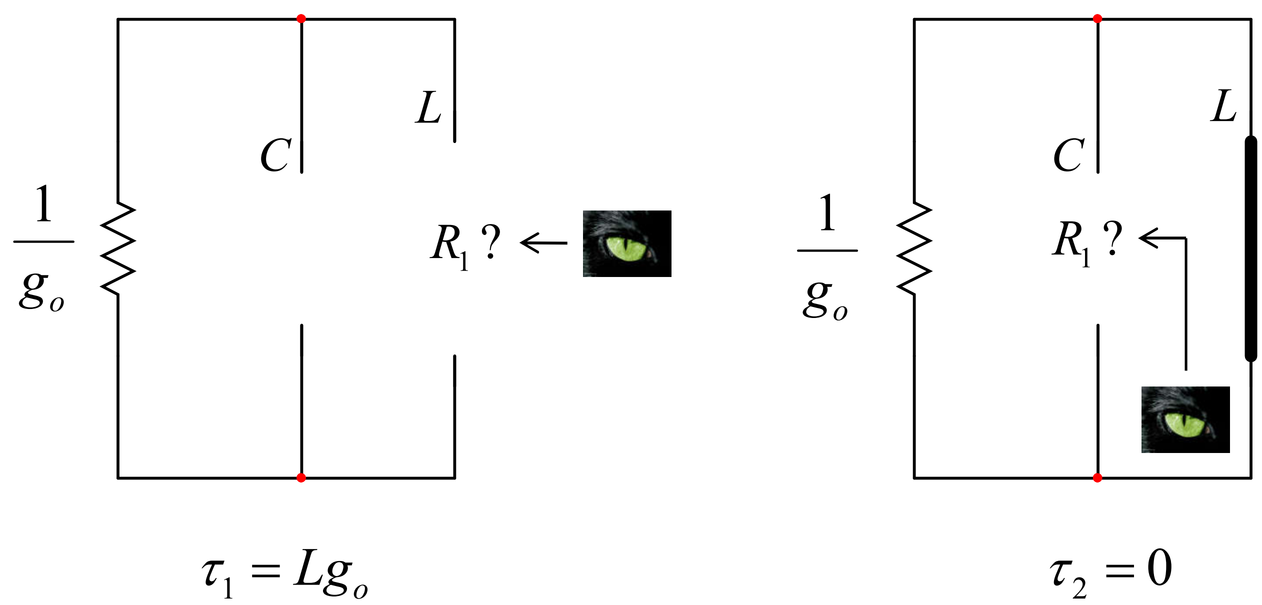


- Circuit approximates current loop dynamics near $\frac{F_{sw}}{2}$

This is a Resonating Tank

- ❑ For a_1 look at the resistance R driving L and C

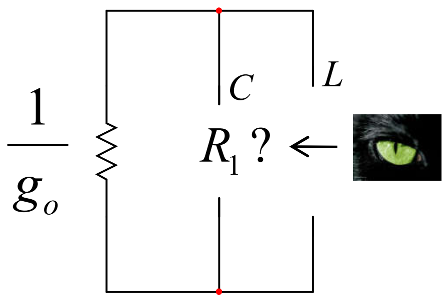
- ➢ Look at the driving impedance at L while C is in its dc state
- ➢ Look at the driving impedance at C while L is in its dc state



$$D(s) = 1 + sLg_o + a_2^2 s^2$$

Find the Second-Order Coefficient a_2

- how τ_1 (involving L) combines with τ'_2 (involving C)?
  - Look at the driving impedance at C while L is in its HF state



$$\tau_1\tau'_2 = Lg_o\frac{C}{g_o} = LC$$

$$\tau'_2 = \frac{C}{g_o}$$

- The denominator can thus be expressed as:

$$D(s) = 1 + sLg_o + s^2LC \longrightarrow D(s) = 1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2$$

$$\omega_0 = \frac{\omega_{sw}}{2} \longrightarrow Lg_o = \frac{1}{\omega_0 Q} = \frac{2}{\omega_{sw} Q}$$

$$Q = \frac{2}{\omega_{sw} Lg_o}$$

What Brings Q to Infinity?

- We now substitute the definition of g_o

$$Q = \frac{2}{\omega_{sw} L g_o} = \frac{2}{\frac{2\pi}{T_{sw}} L \frac{T_{sw}}{L}\left(D'\frac{S_a}{S_n} + \frac{1}{2} - D\right)} = \frac{1}{\pi\left(D'\frac{S_a}{S_n} + \frac{1}{2} - D\right)}$$

- This definition matches the one derived by Ray Ridley

$$Q = \frac{1}{\pi\left(m_c D' - 0.5\right)} \qquad m_c = 1 + \frac{S_a}{S_n}$$

- What happens if $S_a = 0$ with a 50% duty ratio?

$$Q = \frac{1}{\pi\left(\frac{1}{2} - D\right)} \xrightarrow{D = 50\%} Q = \infty$$

R. B. Ridley, "A New Continuous-Time Model for Current-Mode Control", IEEE Transactions of Power Electronics, April 1991

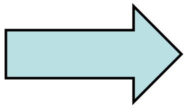
How to Keep the Loop Stable?

- Keeps the denominator away from 0 up to D = 100%

$$D'\frac{S_a}{S_n}+\frac{1}{2}-D\neq 0 \longrightarrow D'\frac{S_a}{S_n}=D-0.5 \xrightarrow[\frac{S_f}{S_n}=\frac{D}{D'}]{D\to 1} \frac{S_n}{S_f}\frac{S_a}{S_n}=D-0.5 \longrightarrow S_a\geq\frac{1}{2}S_f$$

- To prevent overcompensation, damp Q below 1

$$\frac{1}{\pi\left(D'\frac{S_a}{S_n}+\frac{1}{2}-D\right)}<1$$



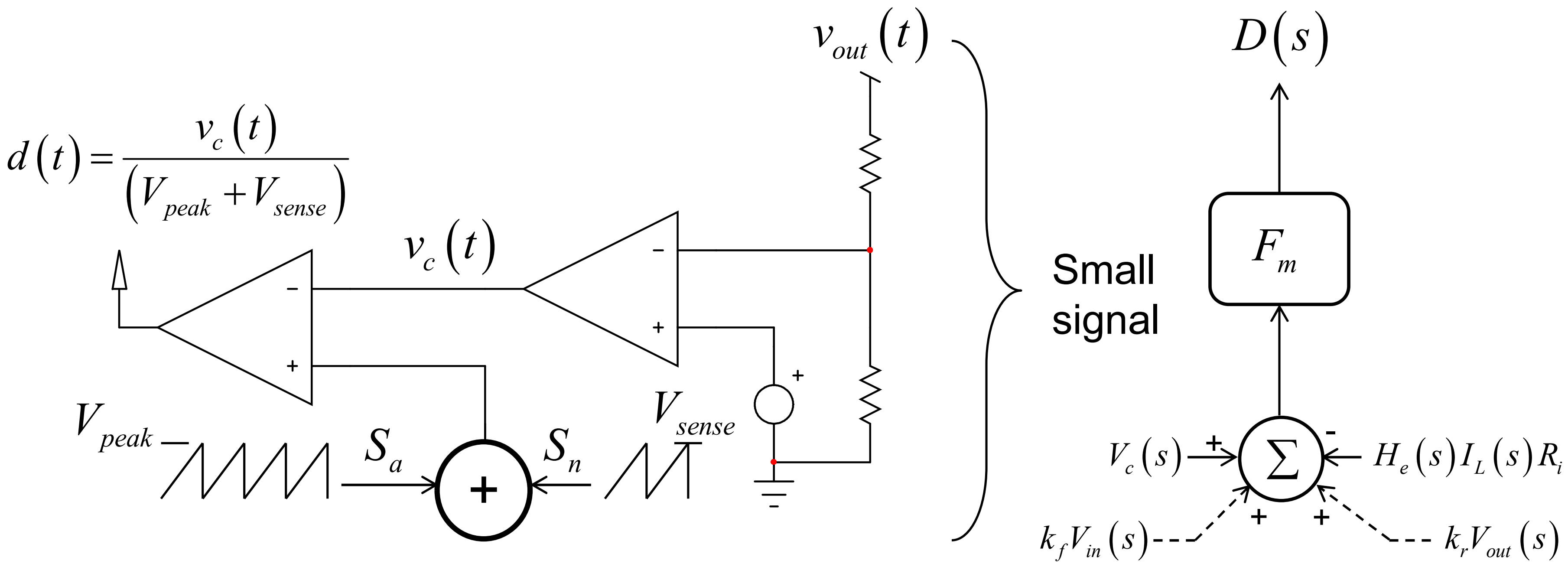
$$S_a\geq\frac{S_n\left(D-0.5+\frac{1}{\pi}\right)}{D'}$$

- Using Ray Ridley's notations, we have:

$$m_c=\frac{\frac{1}{\pi}+0.5}{D'} \longrightarrow 1+\frac{S_a}{S_n}=\frac{\frac{1}{\pi}+0.5}{D'} \longrightarrow S_a\geq\frac{S_n\left(D-0.5+\frac{1}{\pi}\right)}{D'}$$

Do Not Over Compensate the Loop

- ❑ Adding more ramp shifts the control towards voltage mode

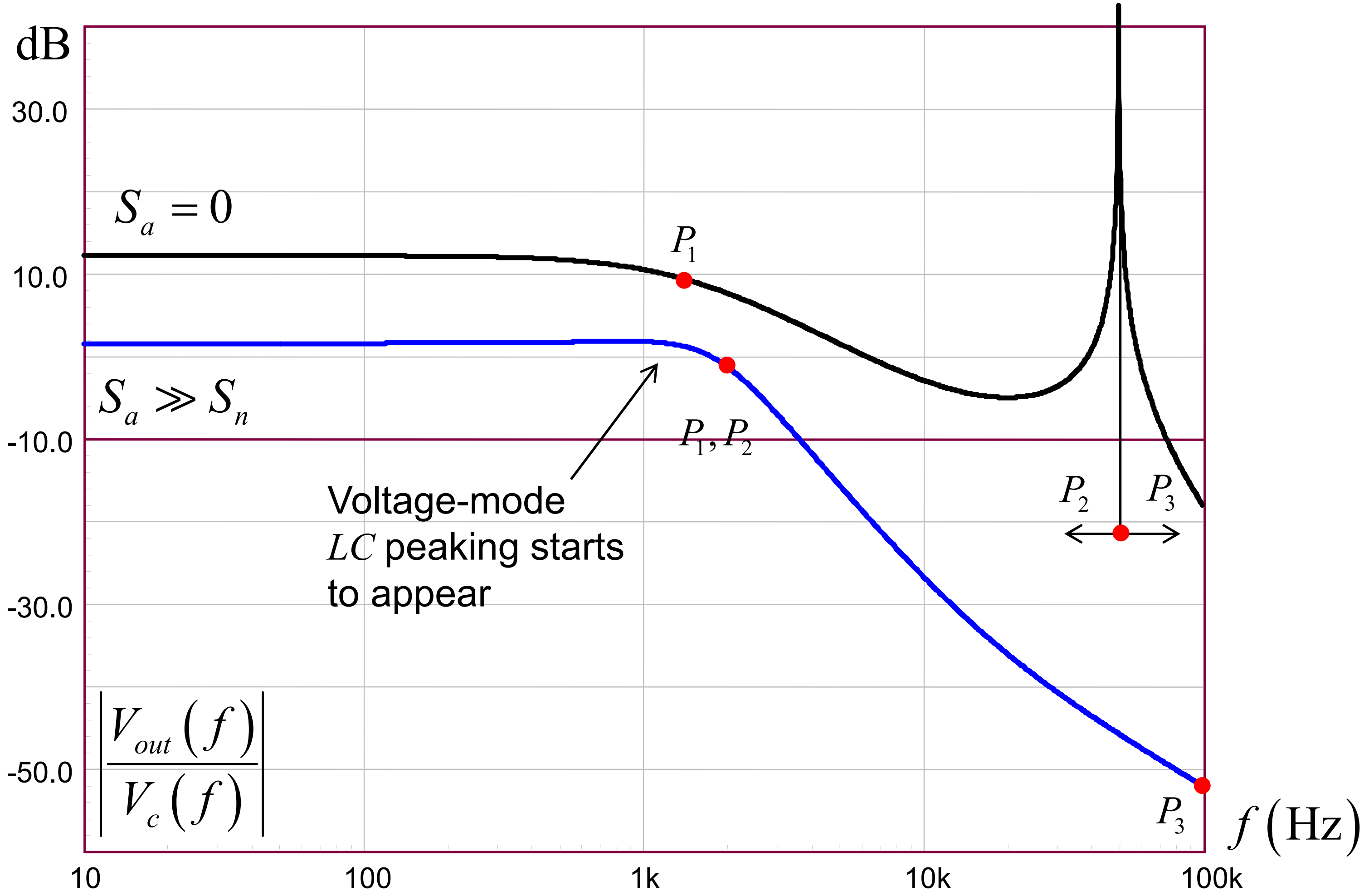


- ❑ The modulator transfer function changes to

$$F_m = \frac{1}{(S_a + S_n)T_{sw}} \xrightarrow{S_a \gg S_n} F_m \approx \frac{1}{S_a T_{sw}} \xrightarrow{S_a = \frac{V_{peak}}{T_{sw}}} G_{PWM} = \frac{1}{V_{peak}}$$

External Ramp Effects on Power Stage

- The double pole splits and joins the low-frequency one



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- ❑ The DCM VM Small-Signal PWM Switch Model
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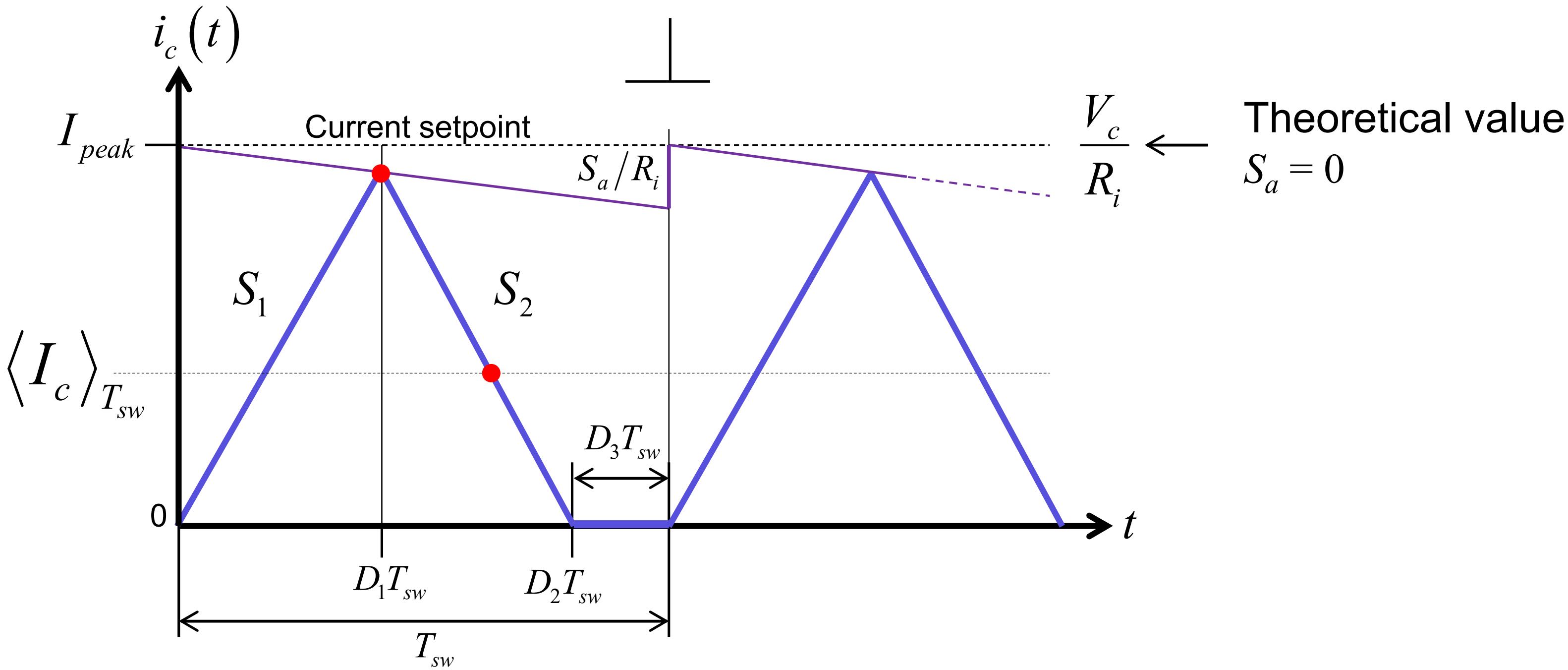
- ❑ The PWM Switch in Boundary Mode

The PWM Switch in DCM

- The PWM CM can work in discontinuous mode

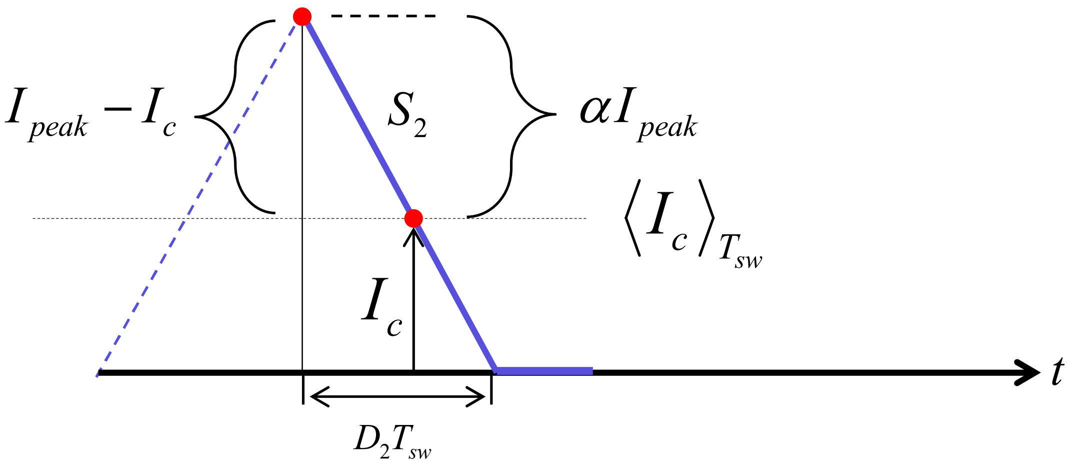


- The average current I_c is somewhere in the downslope S_2

$$I_{peak} = \frac{V_c - D_1 T_{sw} S_a}{R_i} \longrightarrow I_c = \frac{V_c - D_1 T_{sw} S_a}{R_i} - \alpha D_2 T_{sw} S_2$$

Derive the Inductor Average Current

- We must now obtain the value of α to get I_c



- I_c is the area under the triangle divided by the switching period

$$I_c = \frac{I_{peak} D_1}{2} + \frac{I_{peak} D_2}{2} = I_{peak} \frac{D_1 + D_2}{2}$$

$$\alpha I_{peak} = I_{peak} - I_{peak} \frac{D_1 + D_2}{2}$$

$$\alpha = 1 - \frac{D_1 + D_2}{2}$$

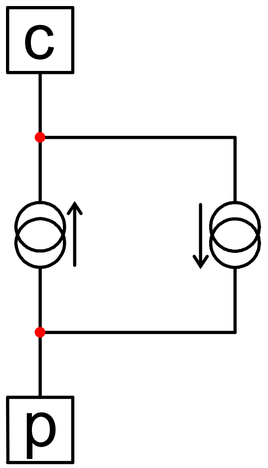
Adopt the CCM Structure for DCM

- Substitute and rearrange to get the inductor current

$$I_c = \frac{V_c}{R_i} - \frac{D_1 T_{sw} S_a}{R_i} - D_2 T_{sw} \frac{V_{cp}}{L}\left(1 - \frac{D_1 + D_2}{2}\right)$$

- If we stick to the original CCM architecture

$$I_c = \frac{V_c}{R_i} - I_\mu \quad \text{with} \quad I_\mu = \frac{D_1 T_{sw} S_a}{R_i} + D_2 T_{sw} \frac{V_{cp}}{L}\left(1 - \frac{D_1 + D_2}{2}\right)$$

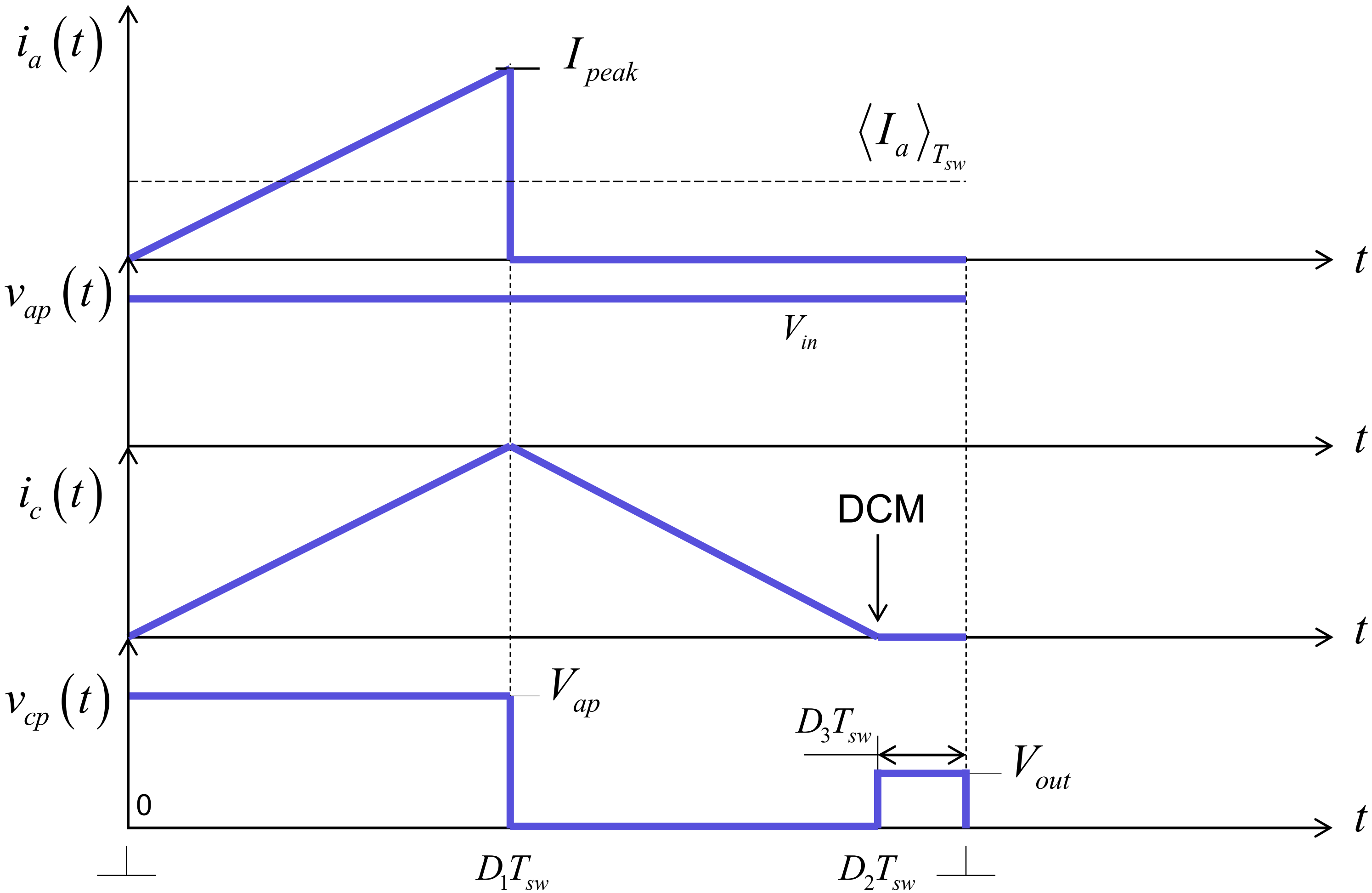


$$\frac{V_c}{R_i}$$

$$\frac{D_1 T_{sw} S_a}{R_i} + D_2 T_{sw} \frac{V_{cp}}{L}\left(1 - \frac{D_1 + D_2}{2}\right)$$

Discontinuous Waveforms

❑ Let's have a look at the PWM switch voltages in DCM



Derive the Duty Ratios

- From the DCM voltage-mode PWM switch we have:

$$V_{cp} = V_{ap}\frac{D_1}{D_1+D_2} \longrightarrow D_1 = \frac{D_2 V_{cp}}{V_{ap}-V_{cp}}$$

- From the operating DCM waveforms

$$I_a = \frac{I_{peak}D_1}{2} \longrightarrow I_{peak} = \frac{2I_a}{D_1} \longrightarrow I_c = I_{peak}\frac{D_1+D_2}{2} \Rightarrow I_a = I_c\frac{D_1}{D_1+D_2}$$

- Almost there, just need to express D_2

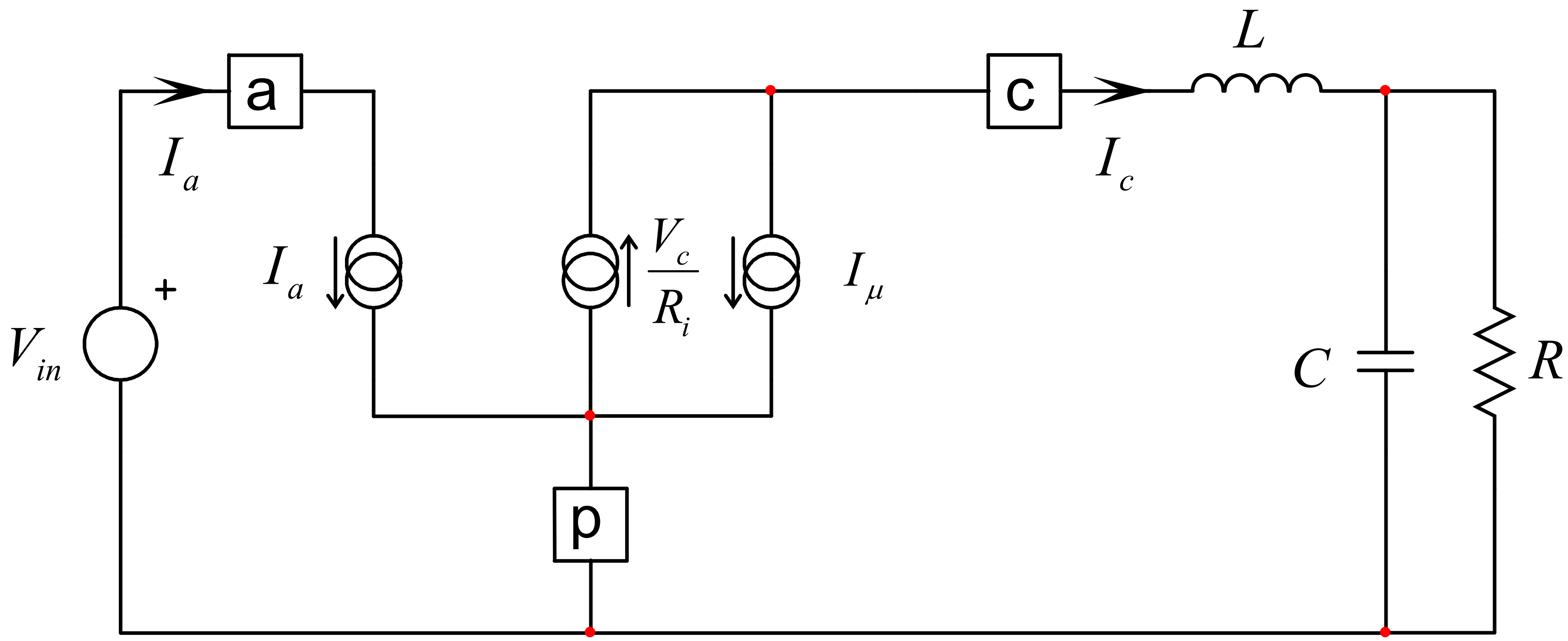
$$I_{peak} = \frac{V_{ac}}{L}D_1T_{sw} \quad \text{and} \quad I_{peak} = \frac{2I_c}{D_1+D_2}$$

$$\frac{V_{ac}}{L}D_1T_{sw} = \frac{2I_c}{D_1+D_2}$$

$$D_2 = \frac{2LF_{sw}}{D_1}\frac{I_c}{V_{ac}} - D_1$$

The DCM Model is Complete!

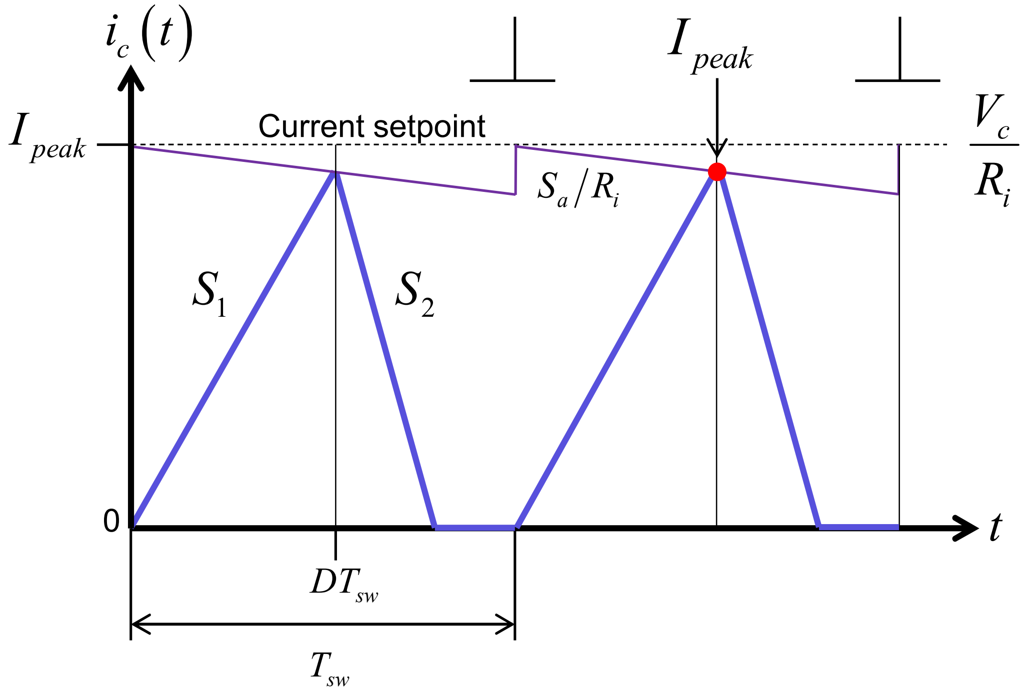
- We can use this model for DCM simulations



$$I_a = I_c \frac{D_1}{D_1 + D_2} \qquad I_\mu = \frac{D_1 T_{sw} S_a}{R_i} + D_2 T_{sw} \frac{V_{cp}}{L}\left(1 - \frac{D_1 + D_2}{2}\right)$$

Another Way of Modeling

❑ We can build a duty ratio factory and use the VM model



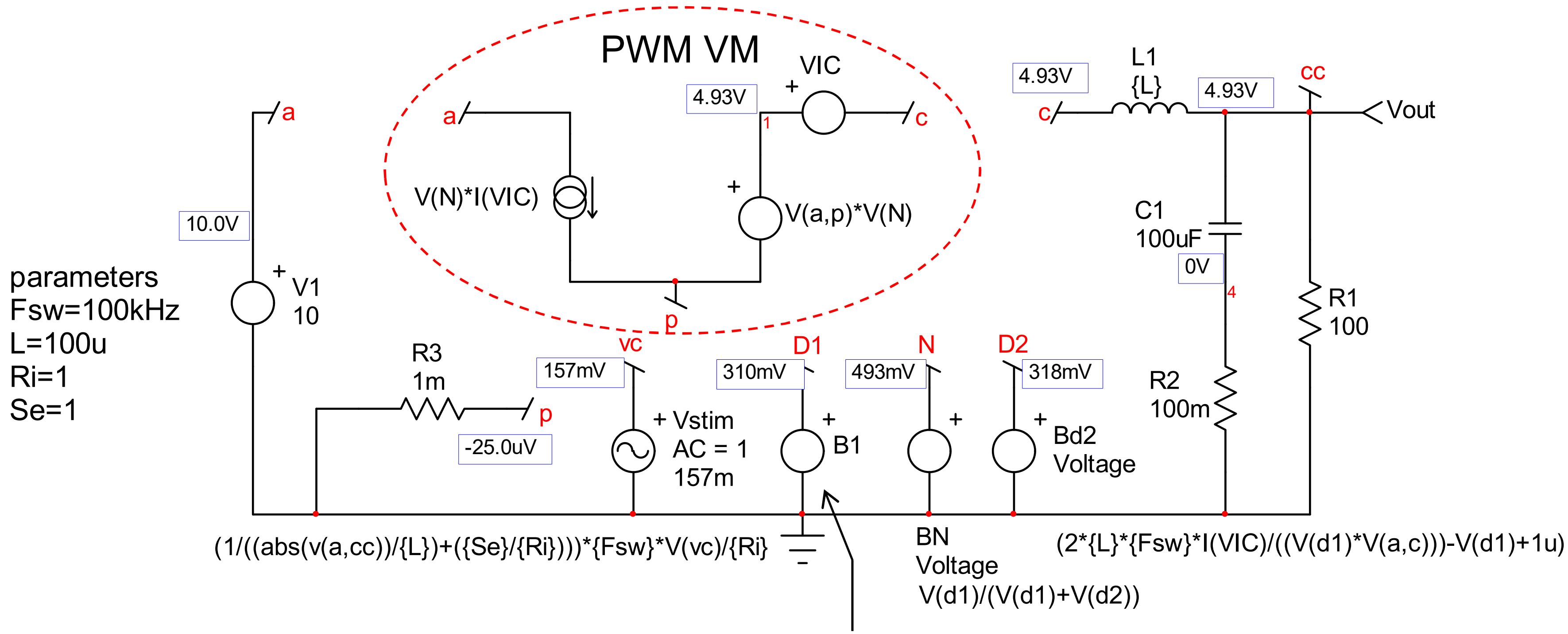
$$I_{peak} = \frac{V_c}{R_i} - \frac{S_a}{R_i} DT_{sw}$$

$$I_{peak} = \frac{V_{ac}}{L} DT_{sw}$$

$$\Rightarrow \quad D = \frac{F_{sw} V_c}{R_i} \frac{1}{\left(\frac{V_{ac}}{L} + \frac{S_a}{R_i} \right)}$$

Build a Duty Ratio Factory

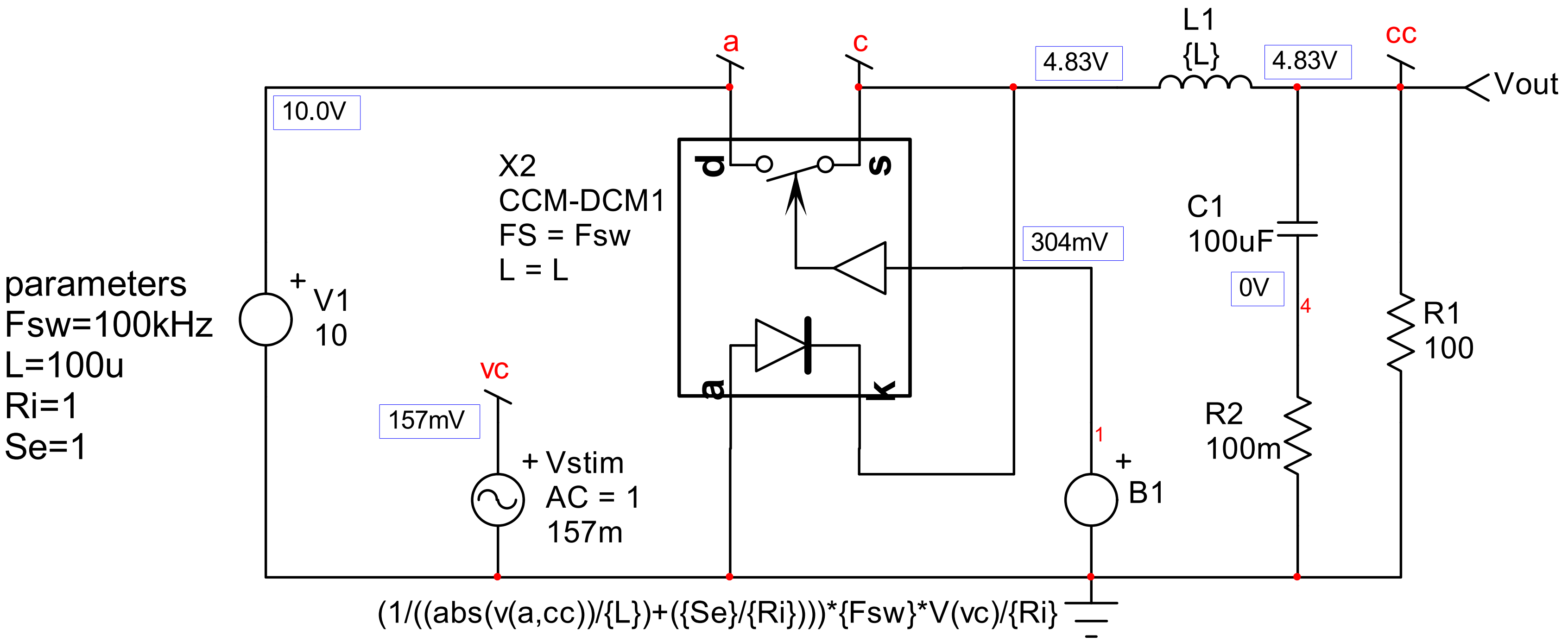
- An in-line equation builds D from the control voltage V_c



Duty ratio factory

Build a Duty Ratio Factory

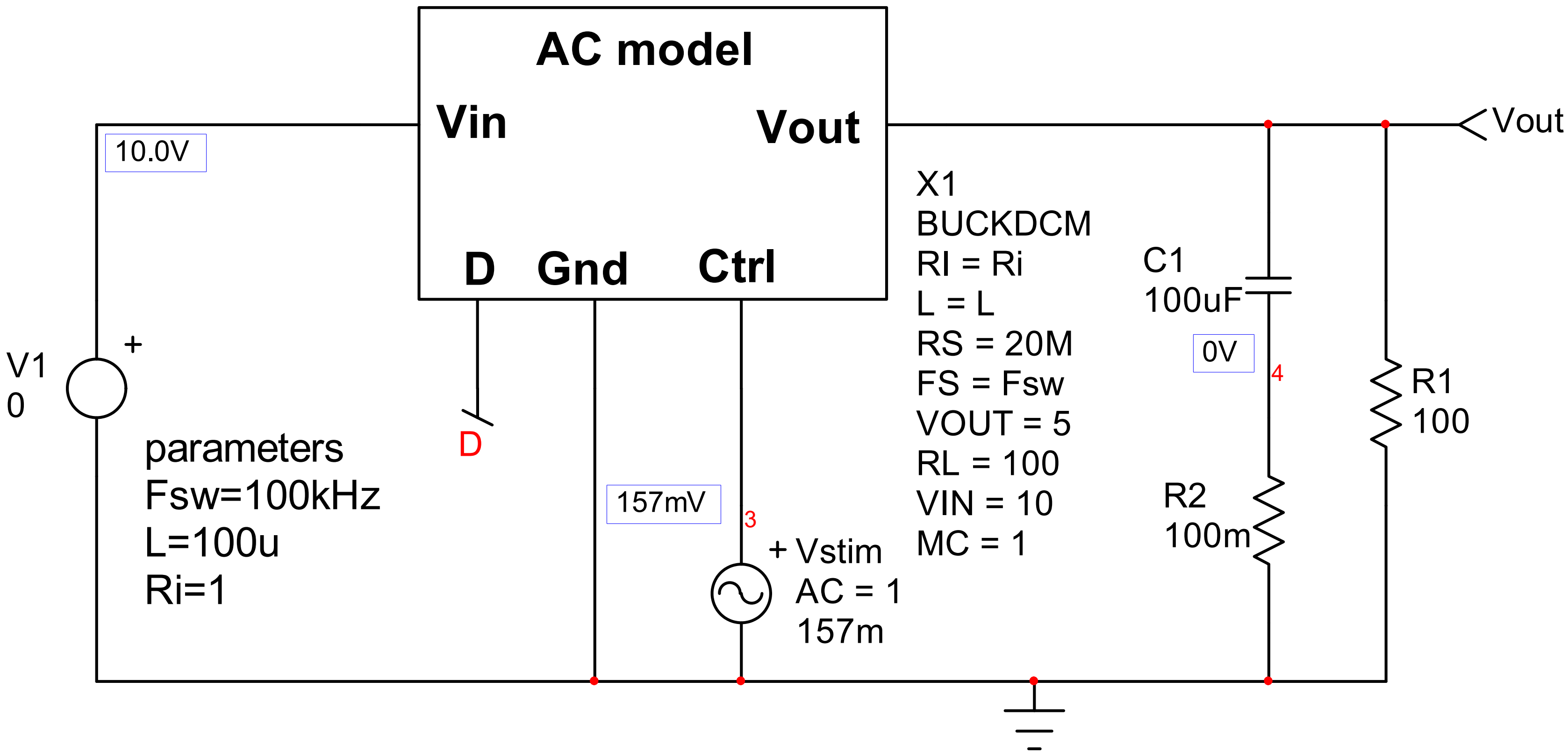
- ❑ You can apply the technique to another VM model



- ❖ This is the auto-toggling CoPEC model

Check the DCM Model Results

- ❑ Check simulation results with Ridley's models



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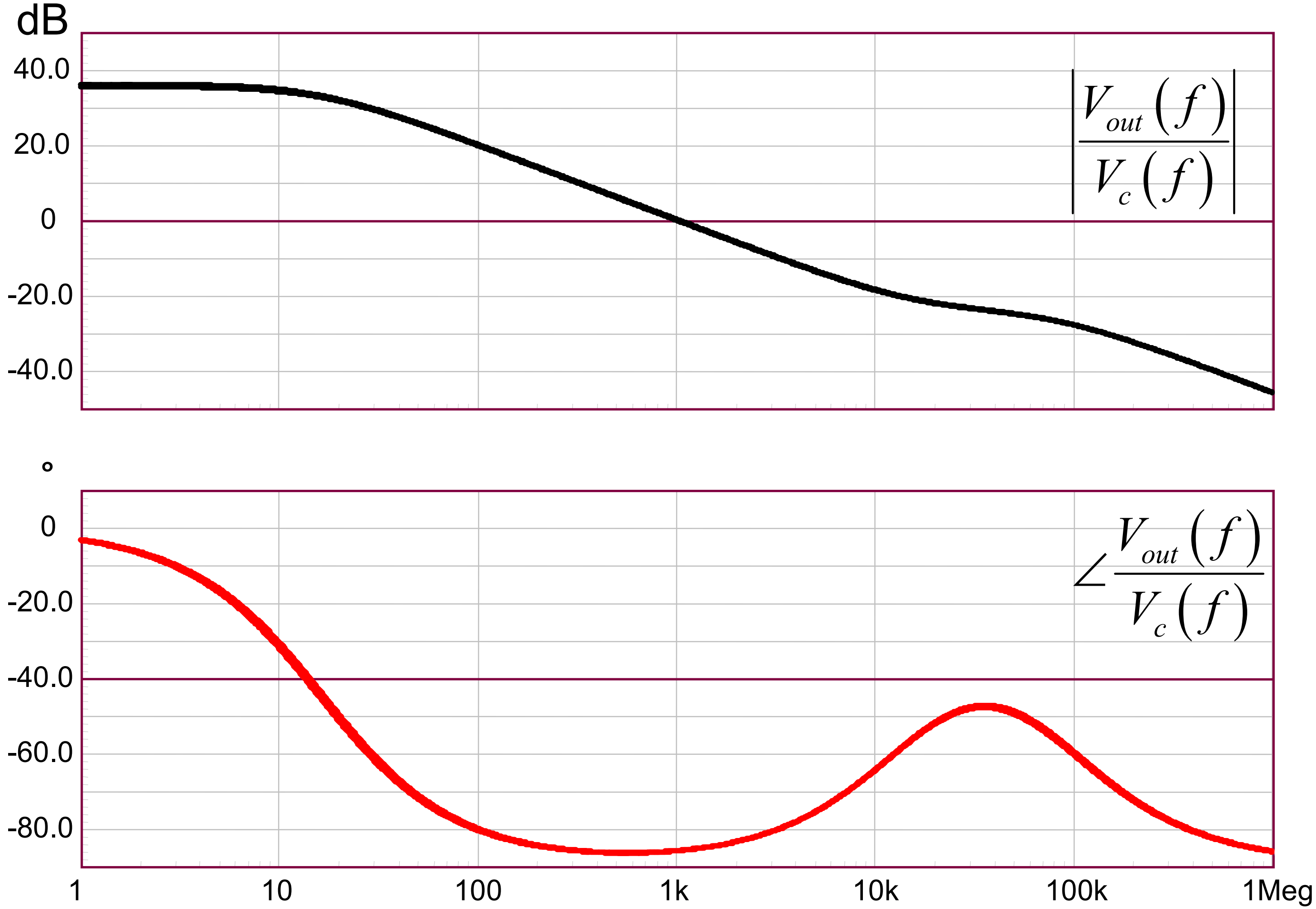
Check the DCM Model Results

- Bode plots are similar along the frequency axis



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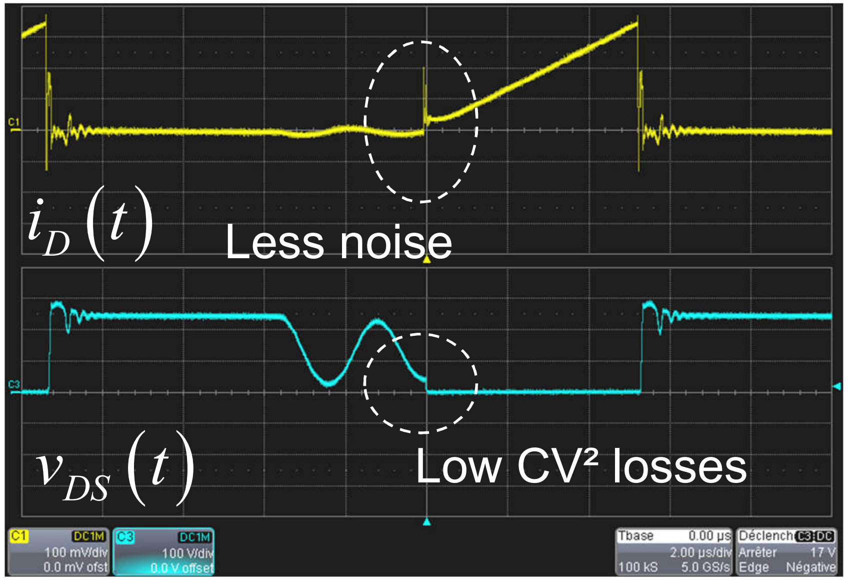
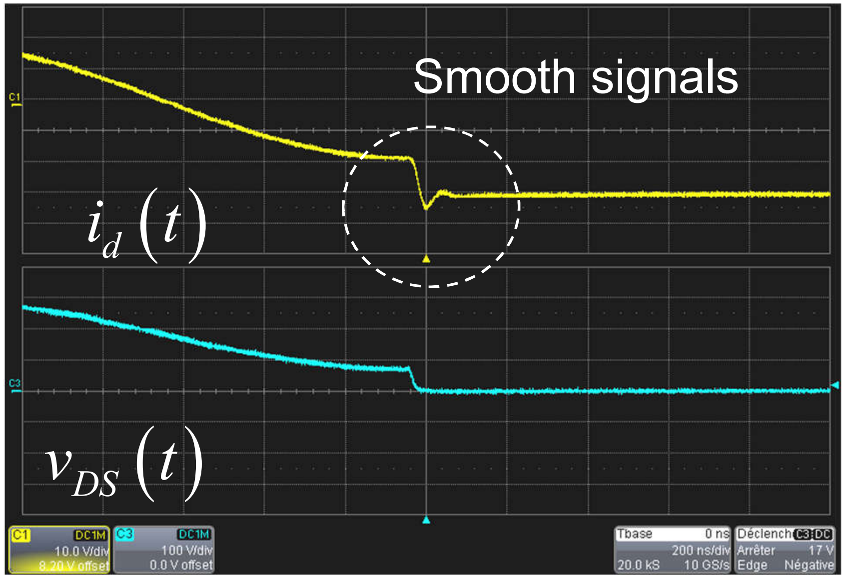
Why QR Operation?

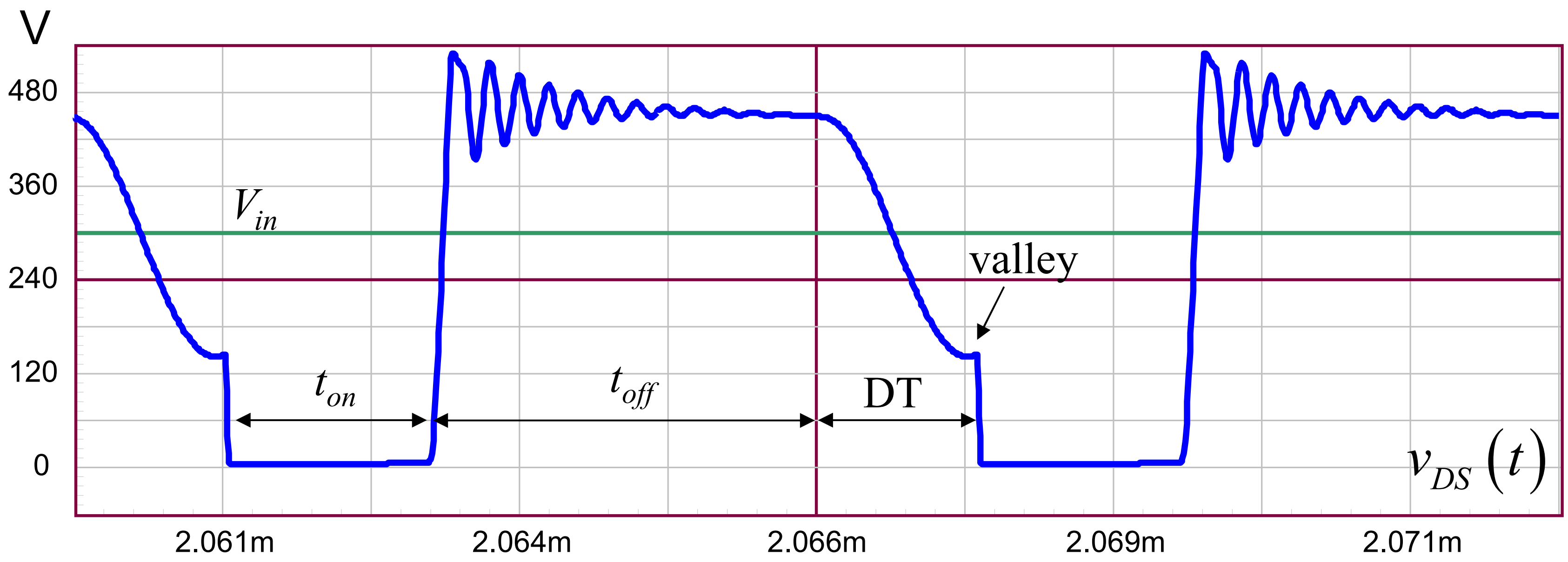
- ❑ More converters are using variable-frequency operation
- ❑ This is known as Quasi-Square Wave Resonant mode: QR
- ➢ Valley switching ensures extremely low capacitive losses
- ➢ DCM operation saves losses in the secondary-side diode
- ➢ Easier synchronous rectification
- ➢ The Right Half-Plane Zero is pushed to high frequencies





What is the Principle of Operation?

- ❑ The drain-source signal is made of peaks and valleys
- ❑ A valley presence means:
  - ➢ The drain is at a minimum level, capacitors are naturally discharged
  - ➢ The converter is operating in the discontinuous conduction mode

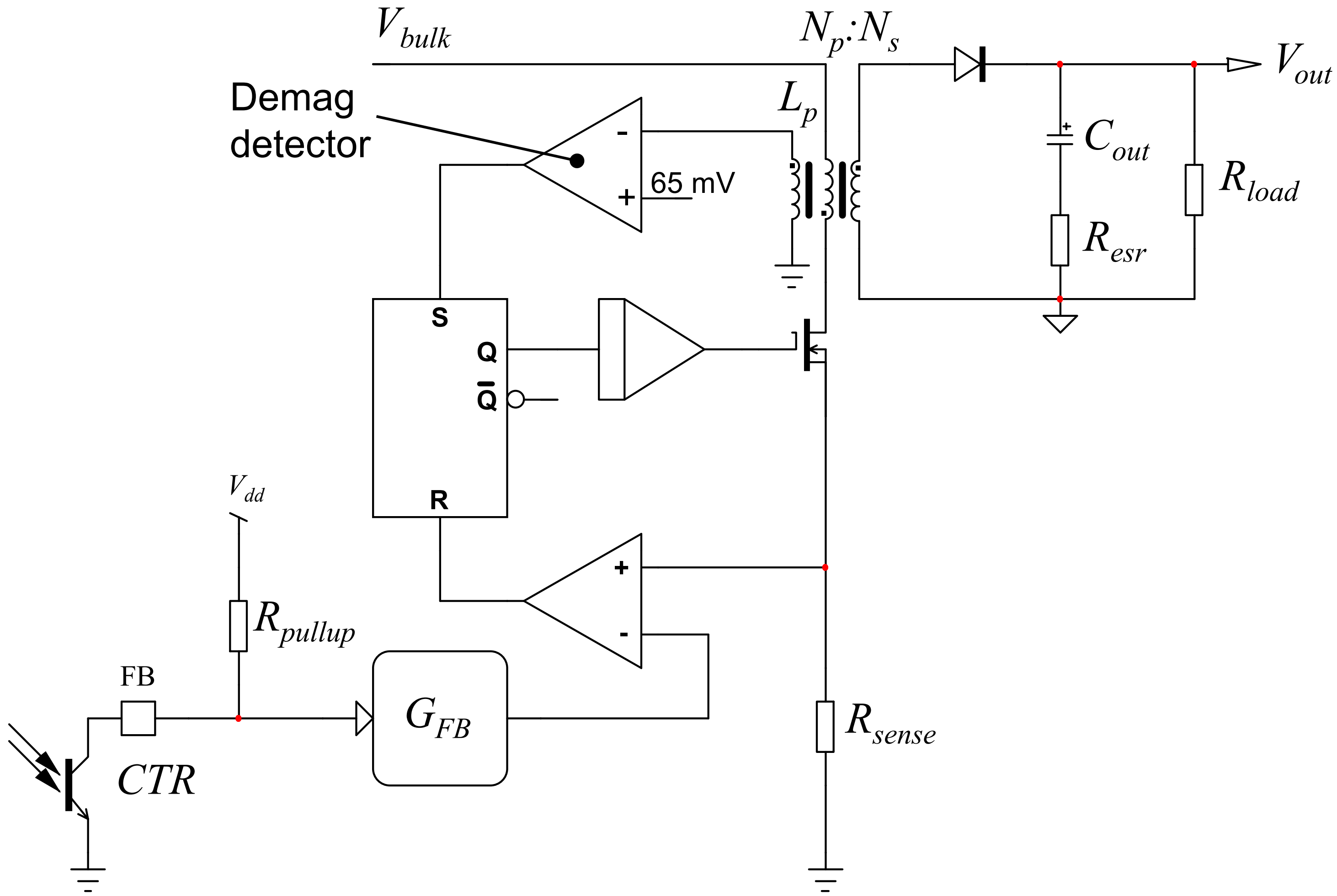


BCM = Borderline or Boundary Conduction Mode

Flyback structure

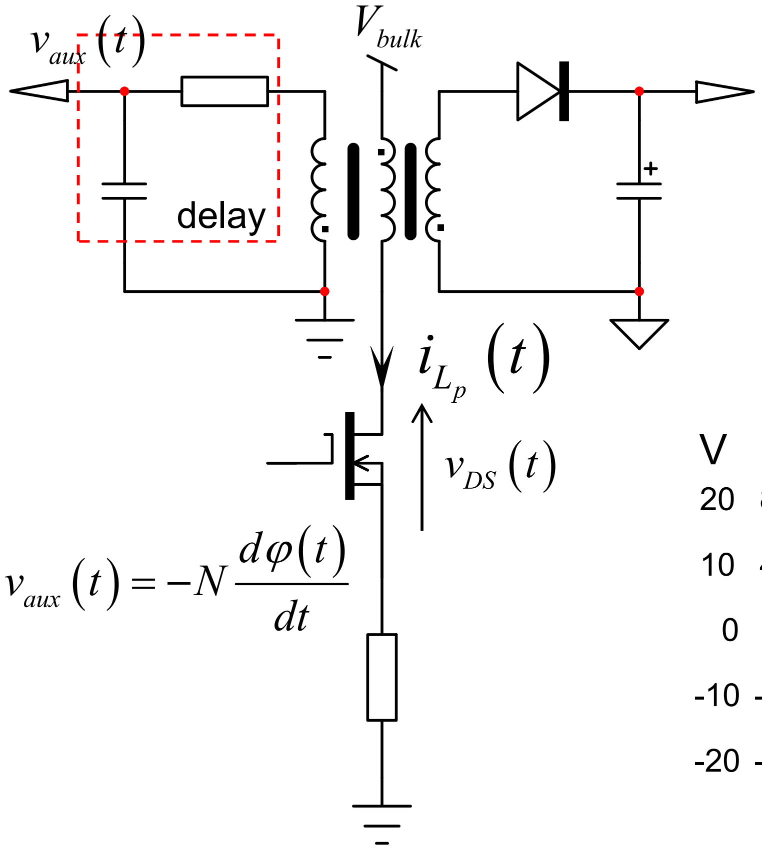
A QR Circuit Does not Need a Clock

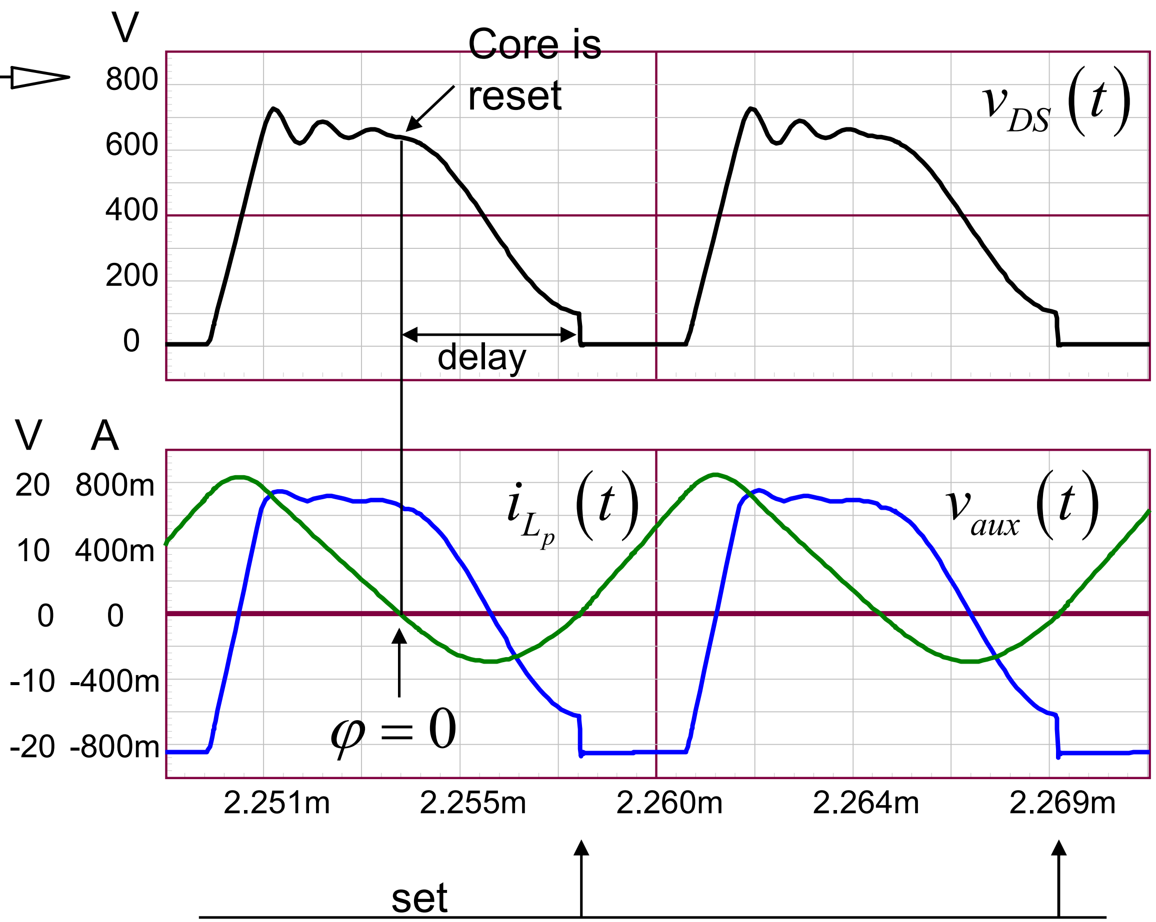
- The system is a self-oscillating current-mode converter



A Winding is Used to Detect Core Reset

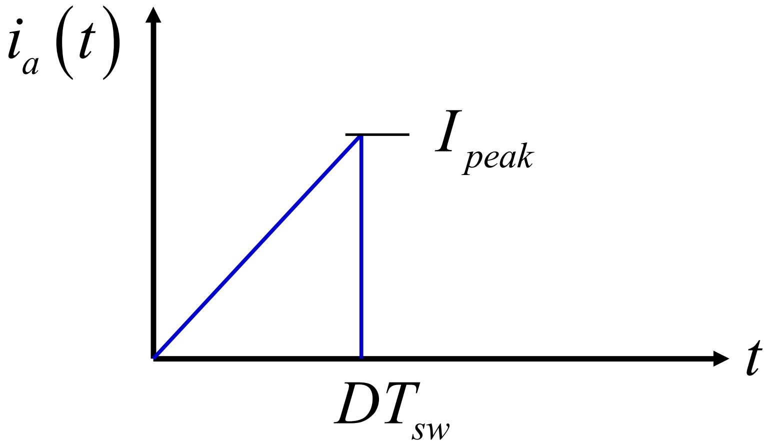
- When the flux returns to zero, the aux. voltage drops
- Discontinuous mode is always maintained

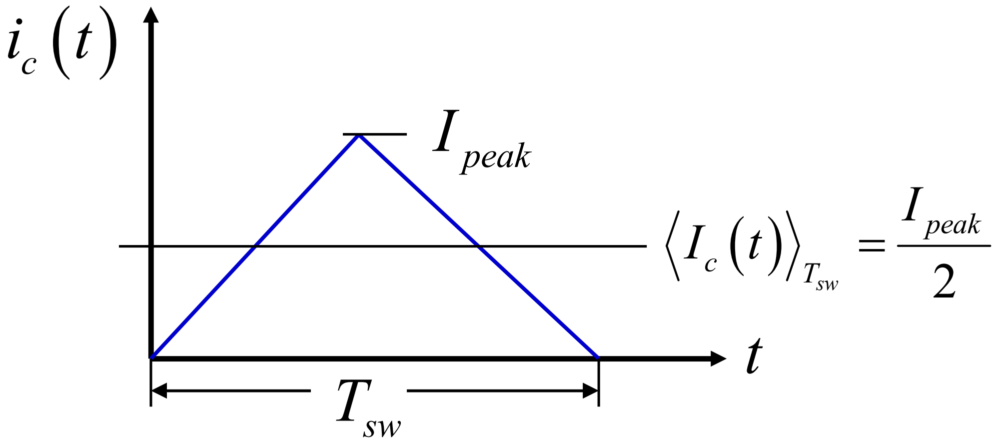




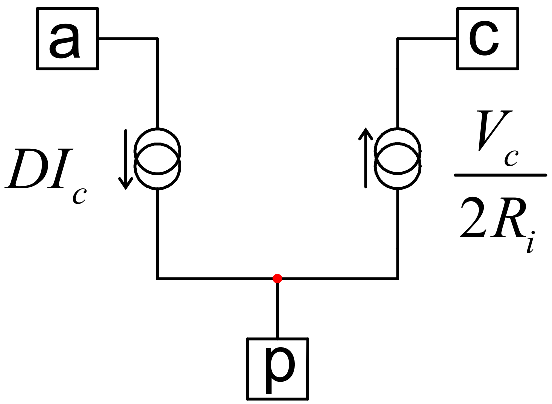
Variable-Frequency Current-Mode

- Observing the waveforms helps us deriving an average model



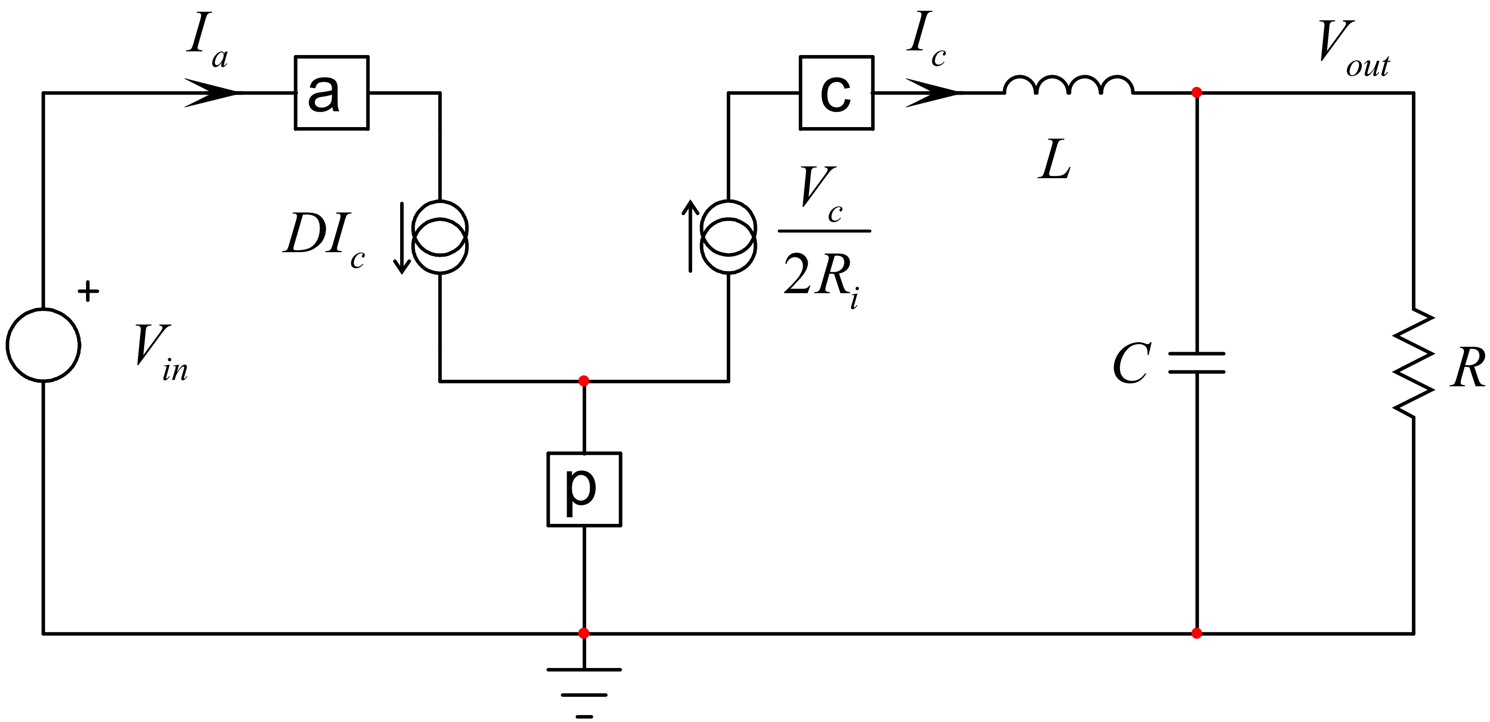


- The structure is similar to that of the current-mode model



Derive the Operating Point

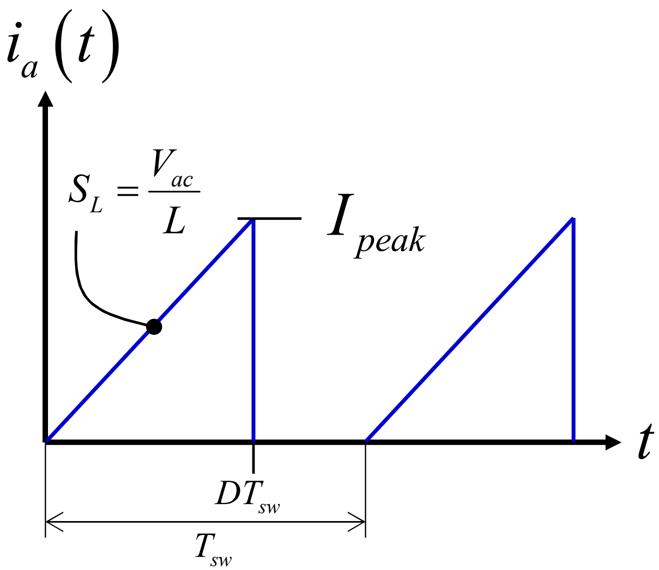
❑ Insert the model in a simple buck converter application



❑ Derive the dc operating point: short inductors, open caps

Write dc Equations

- The inductor peak current is discontinuous



$$I_{peak} = \frac{V_c}{R_i} \qquad I_{peak} = S_L t_{on} = \frac{V_{ac}}{L} DT_{sw}$$

$$\Rightarrow \quad D = \frac{V_c}{R_i} \frac{L}{V_{ac} T_{sw}}$$

- Derive the switching frequency expression

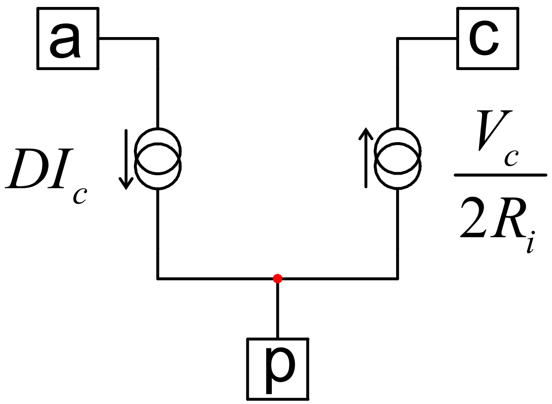
$$t_{on} = \frac{V_c}{R_i} \frac{L}{V_{ac}} \qquad t_{on} = \frac{V_c}{R_i} \frac{L}{V_{cp}}$$

$$\frac{L}{V_{ac}} \rightarrow 1/S_{on} \qquad \frac{L}{V_{cp}} \rightarrow 1/S_{off}$$

$$\Rightarrow \quad T_{sw} = t_{on} + t_{off} = \frac{V_c L}{R_i}\left(\frac{1}{V_{ac}} + \frac{1}{V_{cp}}\right)$$

A Large Signal Model

- A Quasi Resonant model is built with the PWM switch model



$$D = \frac{V_c}{R_i}\frac{L}{V_{ac}T_{sw}} \qquad T_{sw} = \frac{V_c L}{R_i}\left(\frac{1}{V_{ac}} + \frac{1}{V_{cp}}\right)$$

$$I_a = \frac{V_c}{R_i}\frac{L}{V_{ac}T_{sw}} I_c = \frac{I_c}{V_{ac}\left(\frac{1}{V_{ac}} + \frac{1}{V_{cp}}\right)}$$ 3 variables

$$I_c = \frac{V_c}{2R_i}$$ 1 variable

- These are large-signal equations that need linearization

$$I_c = f\left(V_c\right) \longrightarrow \hat{i}_c = \frac{\partial I_c\left(V_c\right)}{\partial V_c}\hat{v}_c \qquad \hat{i}_c = \hat{v}_c\left(\frac{1}{2R_i}\right) = \hat{v}_c k_c \qquad k_c = \frac{1}{2R_i}$$

$$\hat{i}_a = \left.\frac{\partial I_a\left(V_{cp}, V_{ac}, I_c\right)}{\partial V_{cp}}\right|_{I_c, V_{ac}} \hat{v}_{cp} + \left.\frac{\partial I_a\left(V_{cp}, V_{ac}, I_c\right)}{\partial I_c}\right|_{V_{ap}, V_{ac}} \hat{i}_c + \left.\frac{\partial I_a\left(V_{cp}, V_{ac}, I_c\right)}{\partial V_{ac}}\right|_{I_c, V_{cp}} \hat{v}_{ac}$$

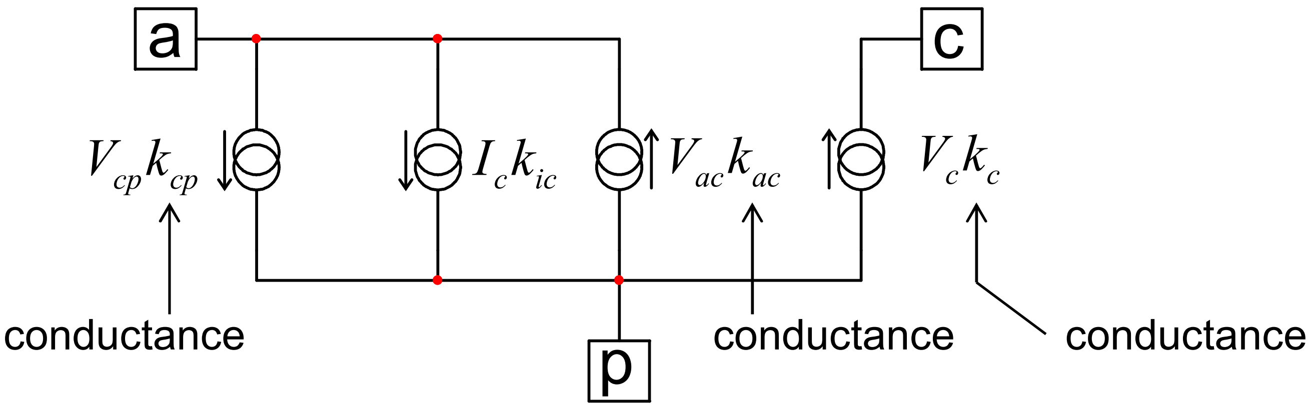
Large to Small-Signal

- Final steps before the small-signal model

$$\hat{i}_a = \hat{v}_{cp}\frac{I_{c0}V_{ac0}}{\left(V_{ac0}+V_{cp0}\right)^2} + \frac{V_{cp0}}{V_{cp0}+V_{ac0}}\hat{i}_c - \hat{v}_{ac}\frac{V_{cp0}I_{c0}}{\left(V_{ac0}+V_{cp0}\right)^2}$$

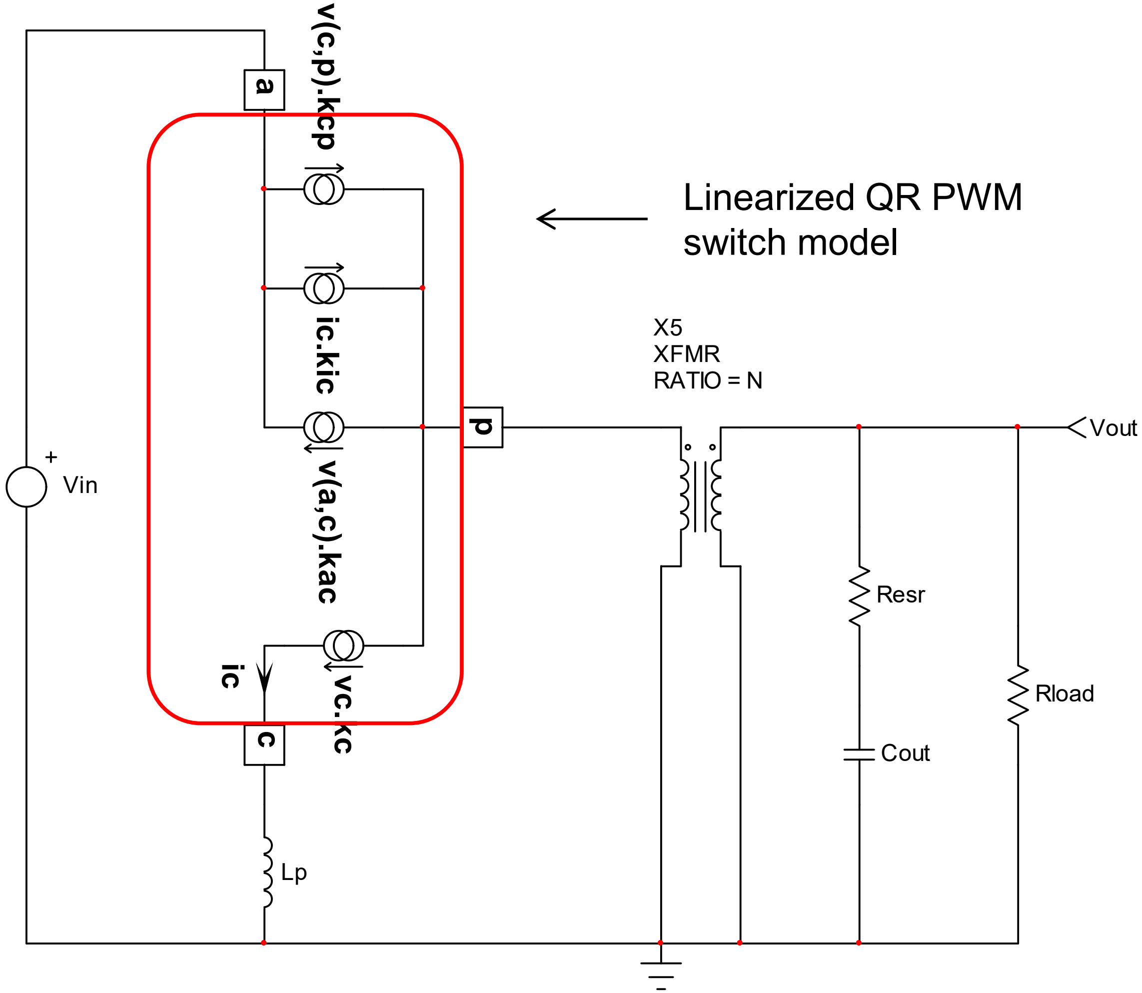
$$k_{cp} = \frac{I_{c0}V_{cp0}}{\left(V_{ac0}+V_{cp0}\right)^2} \qquad k_{ic} = \frac{V_{cp0}}{V_{cp0}+V_{ac0}} \qquad k_{ac} = \frac{V_{cp0}I_{c0}}{\left(V_{ac0}+V_{cp0}\right)^2}$$

- A small-signal model can now be assembled



The Model at Work in an Isolated Converter

- The model can be inserted into a flyback configuration



$$V_{cp0} = \frac{V_{out}}{N} \qquad I_{c0} = \frac{V_{c0}}{2R_i}$$

$$V_{ac0} = V_{in}$$

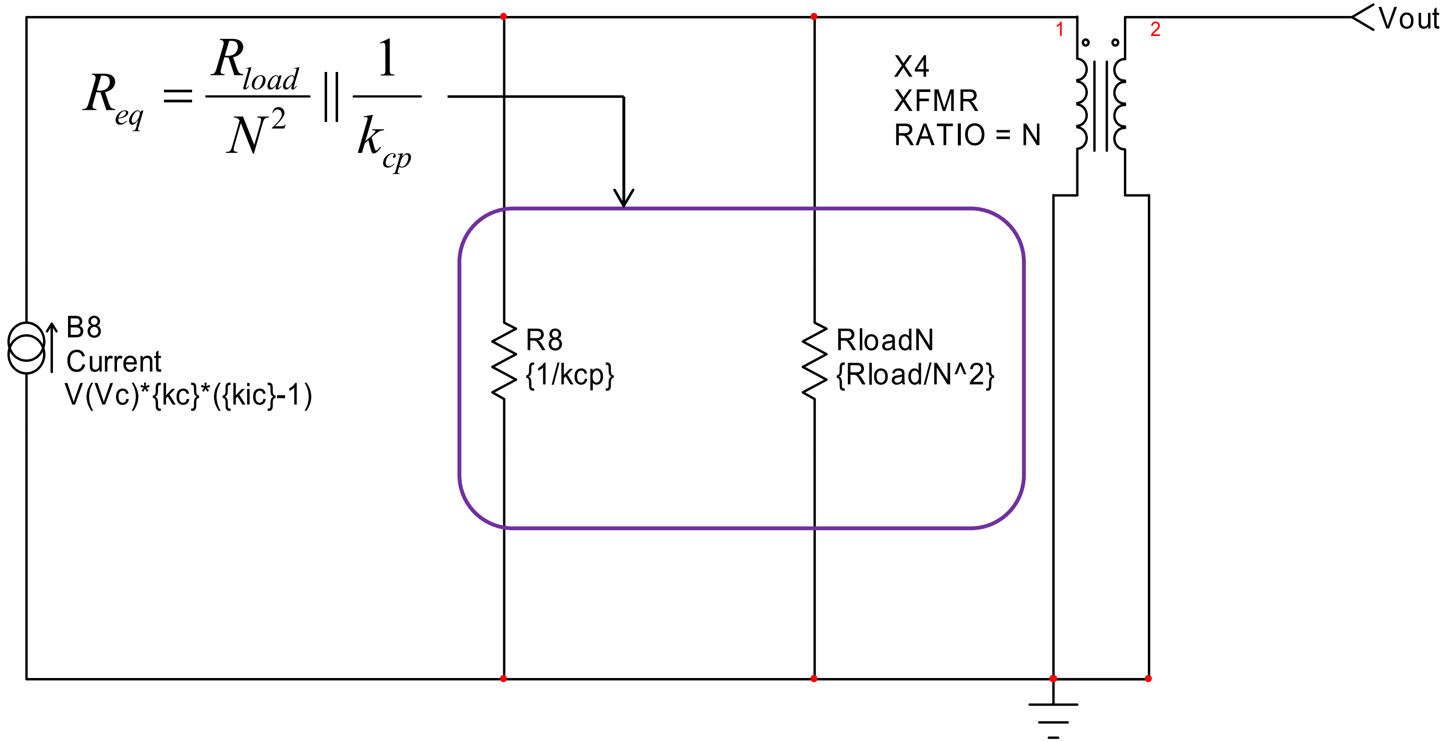
$$k_{cp} = \frac{V_{in} V_{c0} N^2}{2R_i \left(V_{out} + NV_{in}\right)^2}$$

$$k_{ic} = \frac{V_{out}}{V_{out} + NV_{in}}$$

$$k_{ac} = \frac{V_{out} V_{c0} N}{2R_i \left(V_{out} + NV_{in}\right)^2}$$

Start the Study with the dc Gain

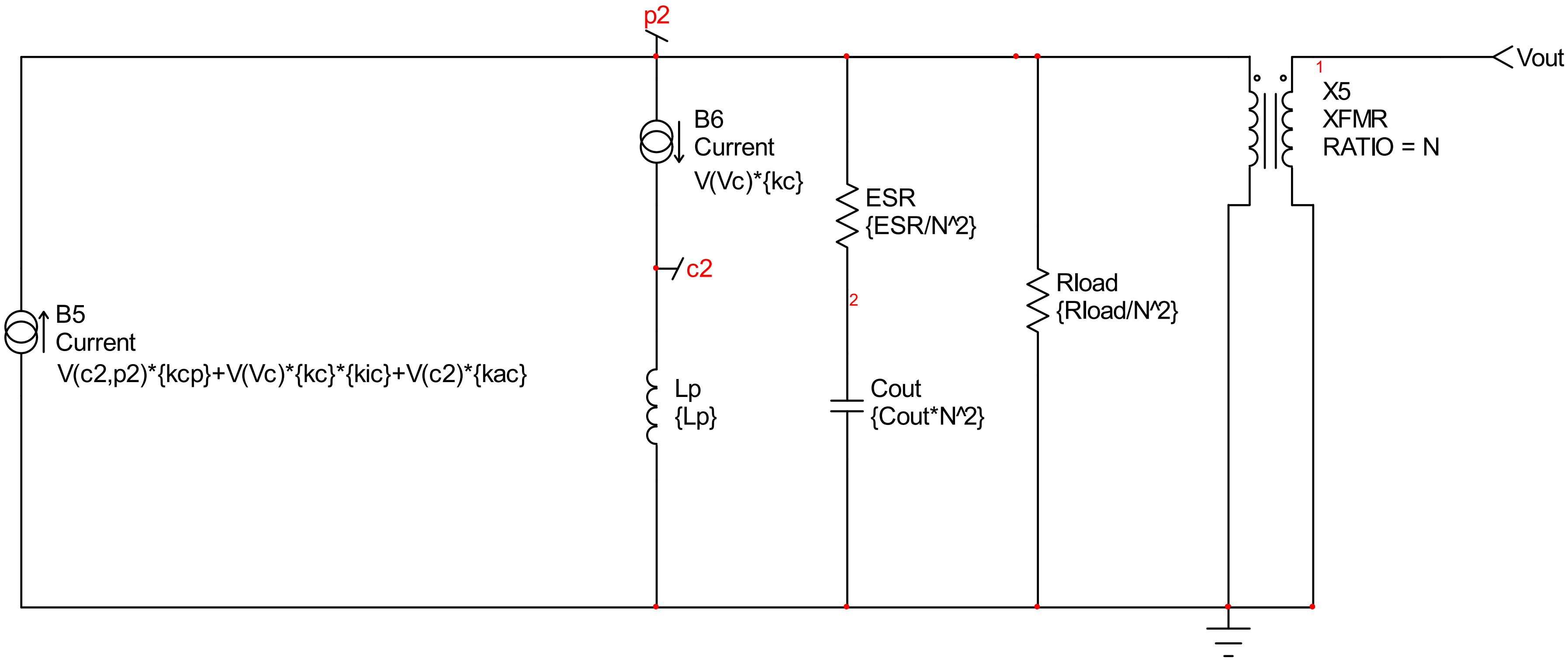
- V_{in} is constant in small-signal, hence $\hat{v}_{in} = 0$



$$G_0 = k_c (1 - k_{ic}) R_{eq} N$$

Put all Elements Back in Place

- The ac analysis requires the complete model



- Kirchhoff's Current and Voltage Laws now apply

Final Transfer Function Equations

- The plant features one Right Half-Plane Zero

$$\frac{\hat{v}_{out}(s)}{\hat{v}_c(s)} = \frac{Nk_c(1-k_{ic})}{k_{cp}+\dfrac{N^2}{R_{load}}} \frac{(1+sR_{ESR}C_{out})\left(1-sL_p\dfrac{k_{cp}+k_{ac}}{1-k_{ic}}\right)}{1+sC_{out}\left(\dfrac{N^2+\dfrac{N^2R_{ESR}}{R_{load}}+k_{cp}R_{ESR}}{k_{cp}+\dfrac{N^2}{R_{load}}}\right)} = G_0\frac{\left(1+\dfrac{s}{s_{z_1}}\right)\left(1-\dfrac{s}{s_{z_2}}\right)}{\left(1+\dfrac{s}{s_{p_1}}\right)}$$

$$G_0 = \frac{Nk_c(1-k_{ic})}{k_{cp}+\dfrac{N^2}{R_{load}}} \qquad s_{z_1} = \frac{1}{R_{ESR}C_{out}} \qquad s_{z_2} = \frac{1-k_{ic}}{(k_{cp}+k_{ac})L_p} = \frac{1}{L_p\dfrac{V_c}{2R_iV_{in}}}$$

$$s_{p_1} = \frac{1}{C_{out}\left(\dfrac{N^2+\dfrac{N^2R_{ESR}}{R_{load}}+k_{cp}R_{ESR}}{k_{cp}+\dfrac{N^2}{R_{load}}}\right)}$$

Check Analytical Results Versus Simulation

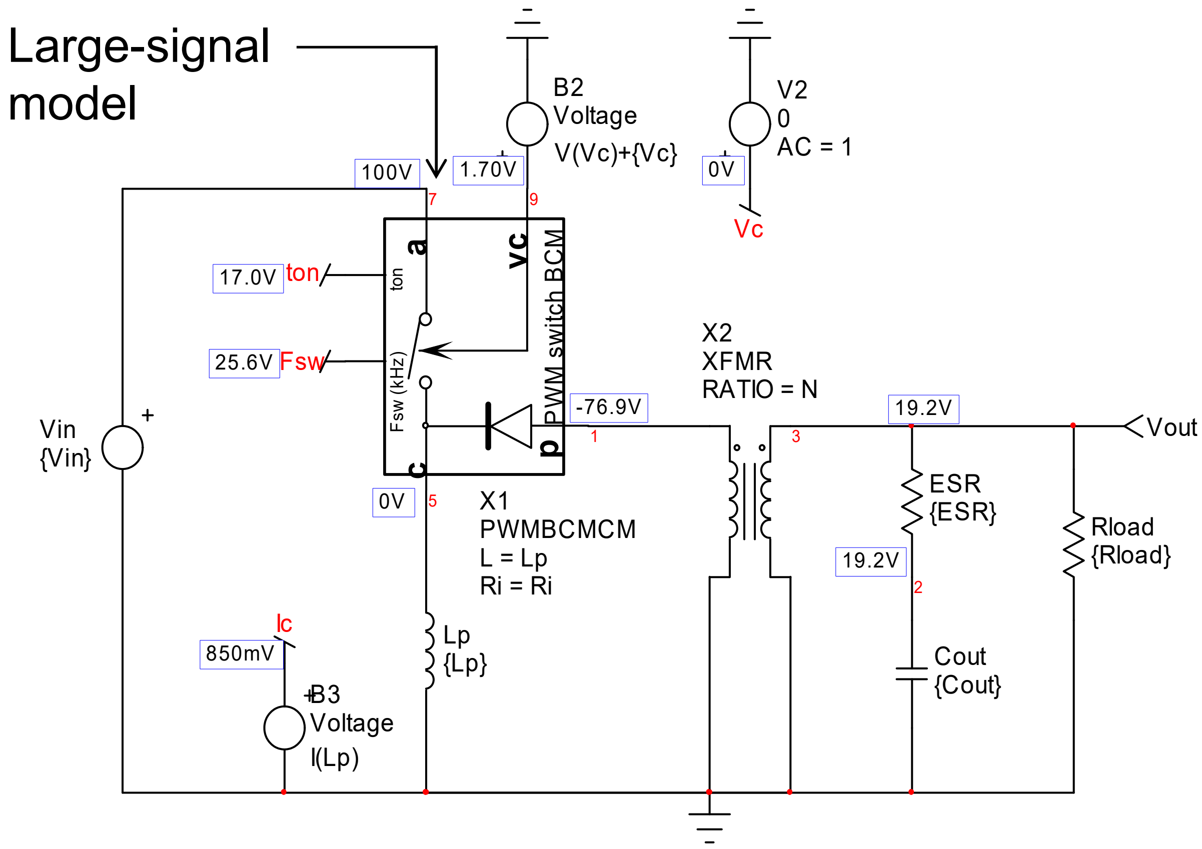
- A sanity check is important to verify the derivation

parameters

Vin=100
Rload=10
N=-0.25
ESR=1
Cout=100u
Lp=1m
Vc=1.7
Ri=1
Fsw=25.6k

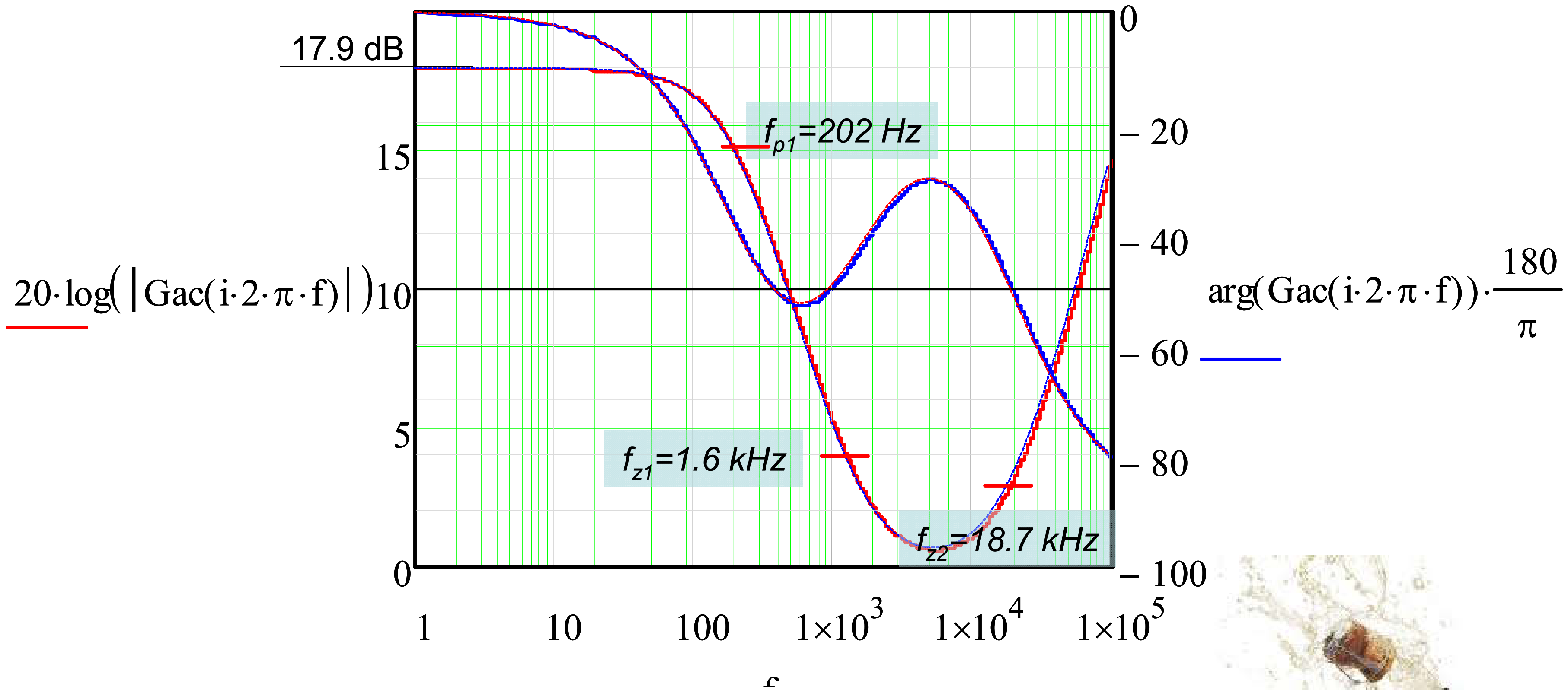
Ic=Vc/(2*Ri)
Vac=100
Vcp=76.9

kc=1/(2*Ri)
kcp=Vac*Ic/(Vcp+Vac)^2
kic=Vcp/(Vcp+Vac)
kac=Vcp*Ic/(Vcp+Vac)^2



Final Lap!

- Compare simulated results with Mathcad® plots



- They perfectly superimpose…

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Conclusion

- ❑ Small-signal modeling is an important part of design
- ❑ Understanding the technique is key to quickly deriving equations
- ❑ State-space averaging is an option but it is tedious and long
- ❑ Small-signal modeling using the PWM switch is simple and fast
- ❑ Available tools help to perform intermediate sanity checks
- ❑ Analytical analysis does not shield you against lab. experiments
- ❑ Analytical analysis, simulation and bench: the best combination!