Introduction to Fast Analytical Techniques: Application to Small-Signal Modeling

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Course Agenda

- What is a Transfer Function?
- Why do We Need New Analytical Techniques?
- Time Constants and Poles
- Identifying the Zeros
- The Null Double Injection
- 2^{nd}-Order Networks
- The PWM Switch Model
- A CCM Buck in Voltage Mode
- A CCM Buck-Boost in Voltage Mode

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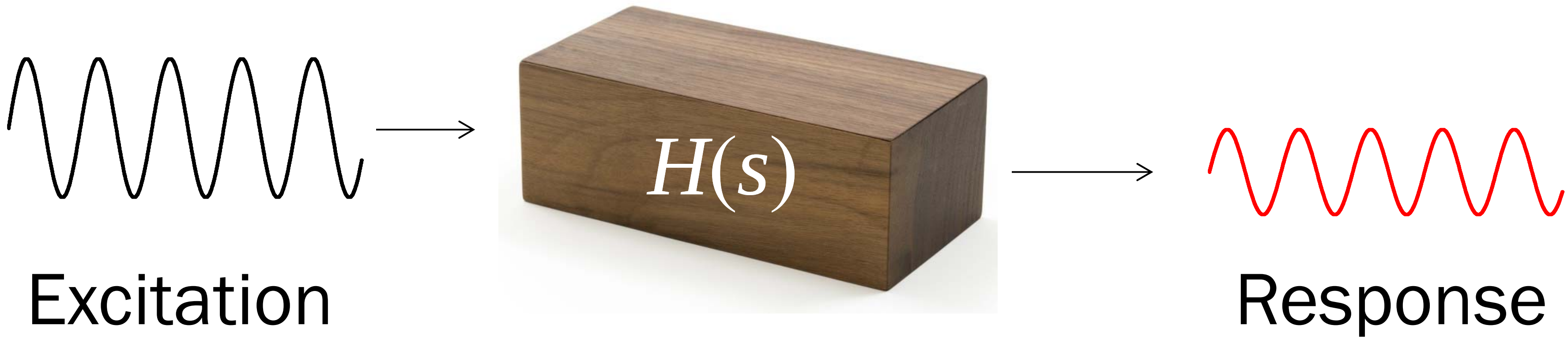
Definition of Transfer Functions

- What is a *transfer function*?

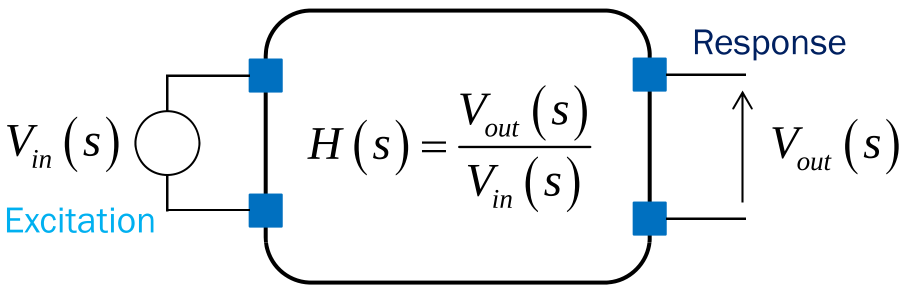


"A transfer function is a mathematical relationship linking a response to an excitation"

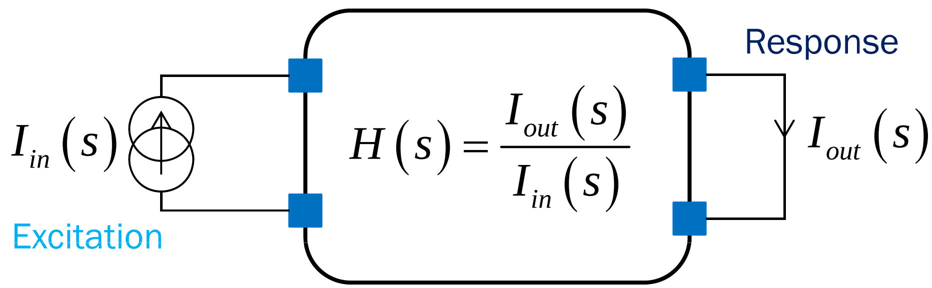
$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} \quad \begin{matrix} \text{response} \\ \text{excitation} \end{matrix}$$

Six Types of Transfer Functions

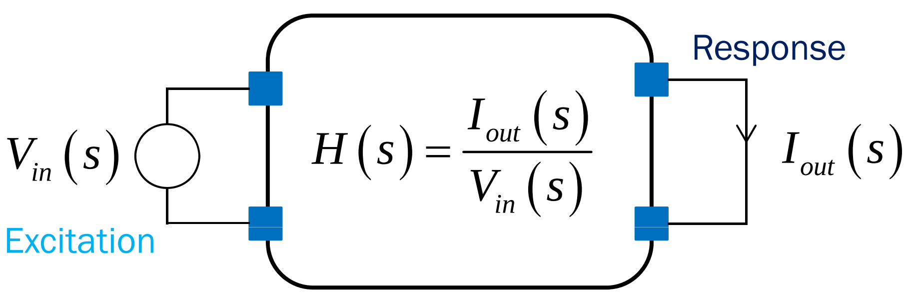
- ❑ Transfer function can involve signals at different places



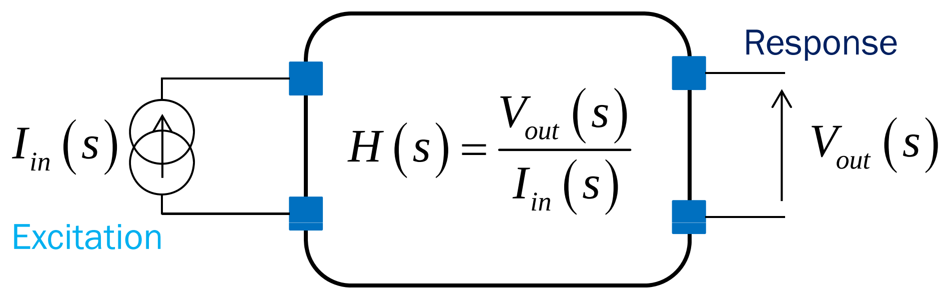
Voltage gain



Current gain



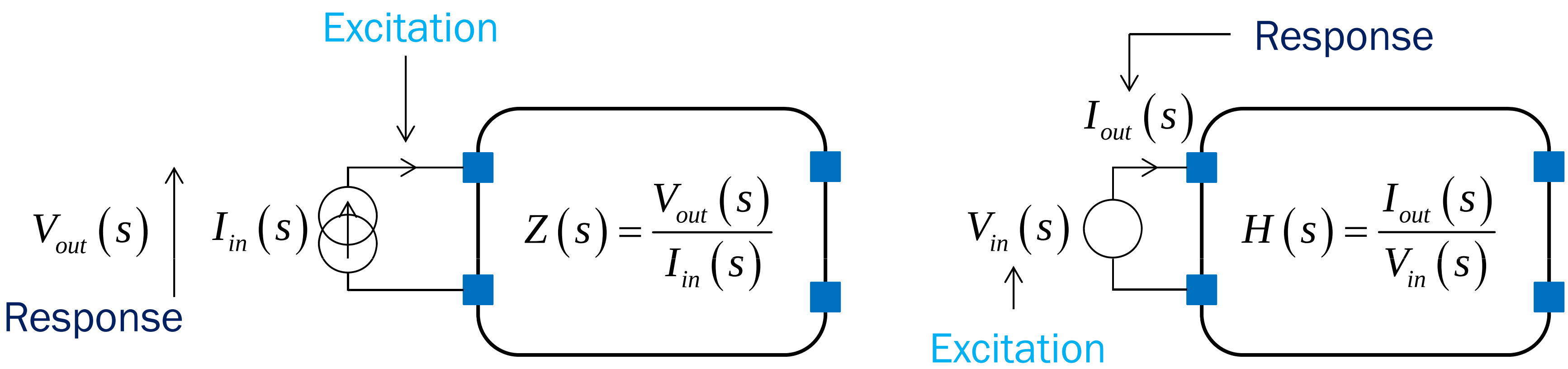
Transfer admittance
or *transadmittance*



Transimpedance

Driving Point Impedance - DPI

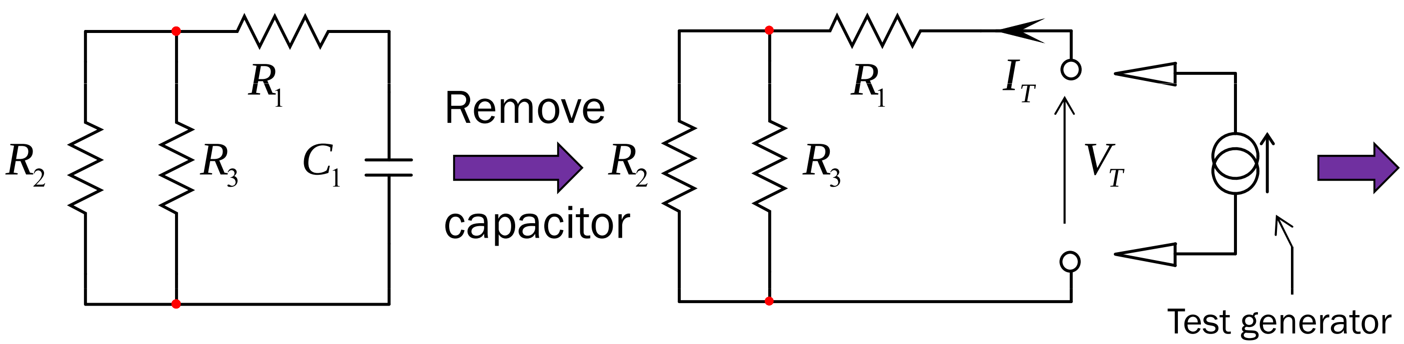
- Waveforms can also be observed at the same terminals



Input impedance

Input admittance

- Determining the resistance at reactance's terminals: DPI



$$R = \frac{V_T}{I_T} = R_2 \parallel R_3 + R_1$$

Writing Transfer Functions

- ❑ How to write a transfer function the right way?
- ✓ A leading term (if any) with the same unit as the function
- ✓ A numerator $N(s)$: its roots are the *zeros* $\left|H\left(s_z\right)\right| = 0$
- ✓ A denominator $D(s)$: its roots are the *poles* $\left|H\left(s_p\right)\right| \to \infty$

gain $$\underset{\text{unitless}}{H}(s) = \underset{\text{unitless}}{H_0}\frac{N(s)}{D(s)}$$

impedance $$\underset{[\Omega]}{Z}(s) = \underset{[\Omega]}{R_0}\underbrace{\frac{N(s)}{D(s)}}_{\text{unitless}}$$

gain $$\underset{[\mathrm{V}]/[\mathrm{V}]}{G}(s) = \underset{[\mathrm{V}]/[\mathrm{V}]}{G_\infty}\frac{N(s)}{D(s)}$$

admittance $$\underset{[\mathrm{S}]}{Y}(s) = \underset{[\mathrm{S}]}{Y_0}\frac{N(s)}{D(s)}$$

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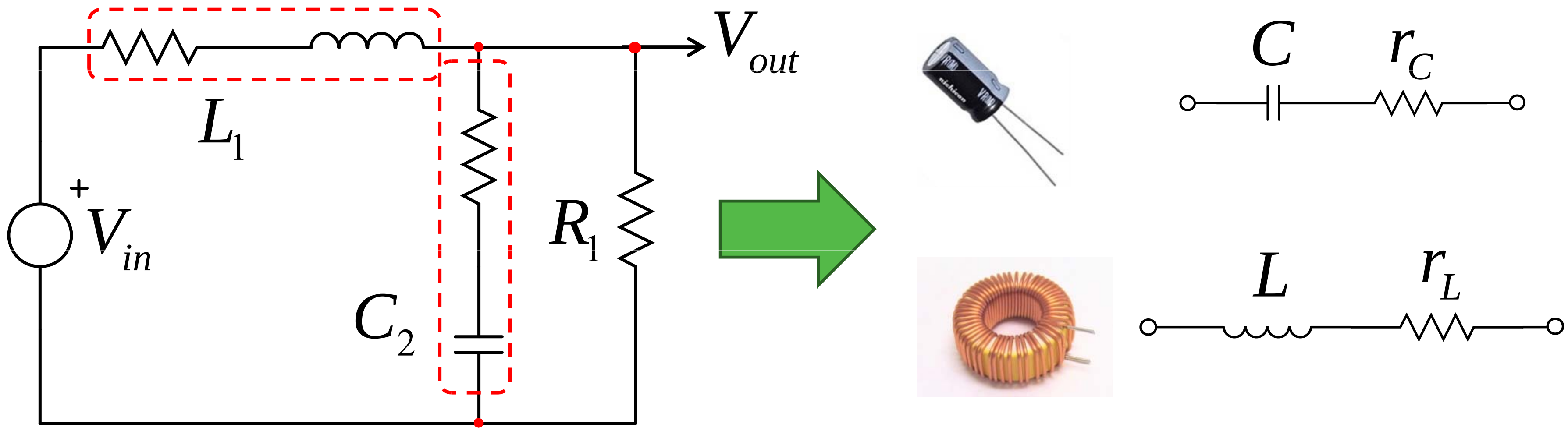
Why a Different Approach?

- ❑ A buck power stage involves energy-storing elements



- ❖ Energy-storing elements host parasitic contributors
- ❖ They move with production, temperature, age…

They hide in the transfer function
and must be unmasked!

Identifying the Contributors

- Brute-force algebra complicates analysis

$$H(s)=\frac{\left(\frac{1}{sC_2}+r_C\right)\|R_1}{\left(\frac{1}{sC_2}+r_C\right)\|R_1+r_L+sL_1}=\frac{\overbrace{\boxed{R_1+sR_1r_CC_2}}^{\text{Zeros?}}}{\underbrace{\boxed{R_1+r_L+sL_1+sC_2R_1r_C+C_2R_1r_Ls+C_2r_Cr_Ls+C_2L_1R_1s^2+C_2L_1r_Cs^2}}_{\text{Poles?}}}$$

Dc gain?

- More energy is needed to unveil these terms
  - factor and rearrange coefficients
  - simplify numerator and denominator
  - Don't make mistakes!

This is a *high-entropy* expression

In thermodynamics, entropy is a measure of disorder, http://en.wikipedia.org/wiki/Entropy

Low-Entropy Expressions

- What if you could write the expression in one shot?

$$H(s)=\boxed{\frac{R_1}{R_1+r_L}}\frac{1+sr_CC_2}{1+\boxed{s\left[\frac{L_1}{r_L+R_1}+C_2\left(r_C+r_L \parallel R_1\right)\right]}+\boxed{s^2L_1C_2\frac{r_C+R_1}{r_L+R_1}}}$$

- Naturally reading gains, poles and zeros...

$$H(s)=\boxed{H_0}\frac{1+\frac{s}{\omega_z}}{1+\boxed{\frac{s}{\omega_0 Q}}+\boxed{\left(\frac{s}{\omega_0}\right)^2}}$$

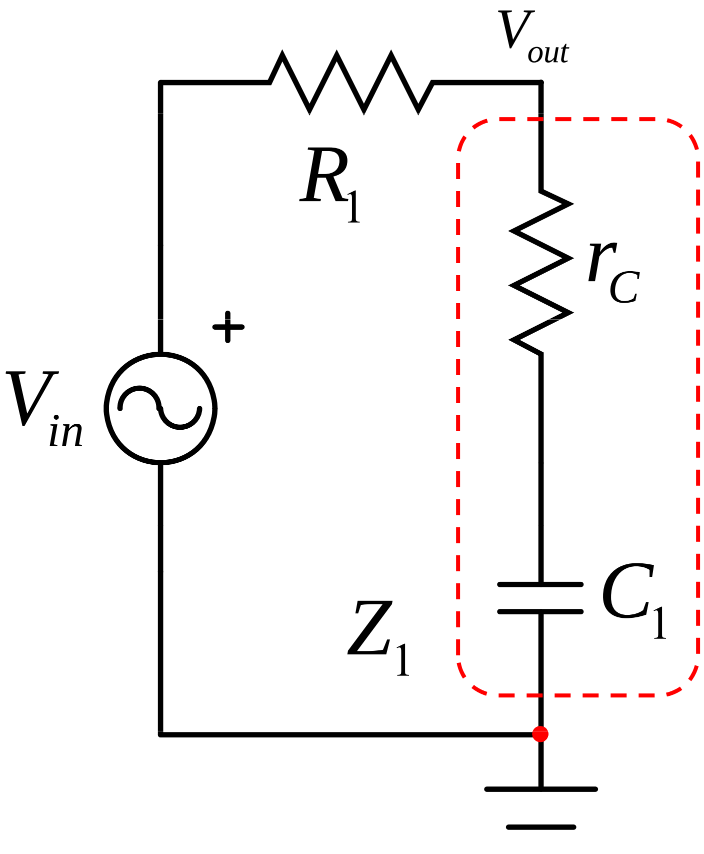
$$\left.\begin{aligned}&\omega_z=\frac{1}{r_CC_2} \qquad \omega_0=\frac{1}{\sqrt{L_1C_2}}\sqrt{\frac{r_C+R_1}{r_L+R_1}}\\&Q=\frac{r_L+R_1}{L_1+C_2\left[r_Cr_L+R_1\left(r_C+r_L\right)\right]}\frac{1}{\omega_0}\end{aligned}\right\}$$

This is a *low-entropy* expression

R. D. Middlebrook, “Methods of Design-Oriented Analysis: Low Entropy Expressions”, New Approaches to Undergraduate Education, July 1992

Starting with a Simple Example

- What is the transfer function of the below circuit?



$$Z_1(s) = r_C + \frac{1}{sC_1} = \frac{1 + s r_C C_1}{sC_1}$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{\dfrac{1 + s r_C C_1}{sC_1}}{\dfrac{1 + s r_C C_1}{sC_1} + R_1}$$

- Is there any easier and faster way to go?

Two Different Stages

- Consider dc and high-frequency states for L and C

C → impedance → $Z_C = \frac{1}{sC}$

Dc state	$Z_C = \infty$	Cap. is an open circuit
HF state	$Z_C = 0$	Cap. is a short circuit

L → impedance → $Z_L = sL$

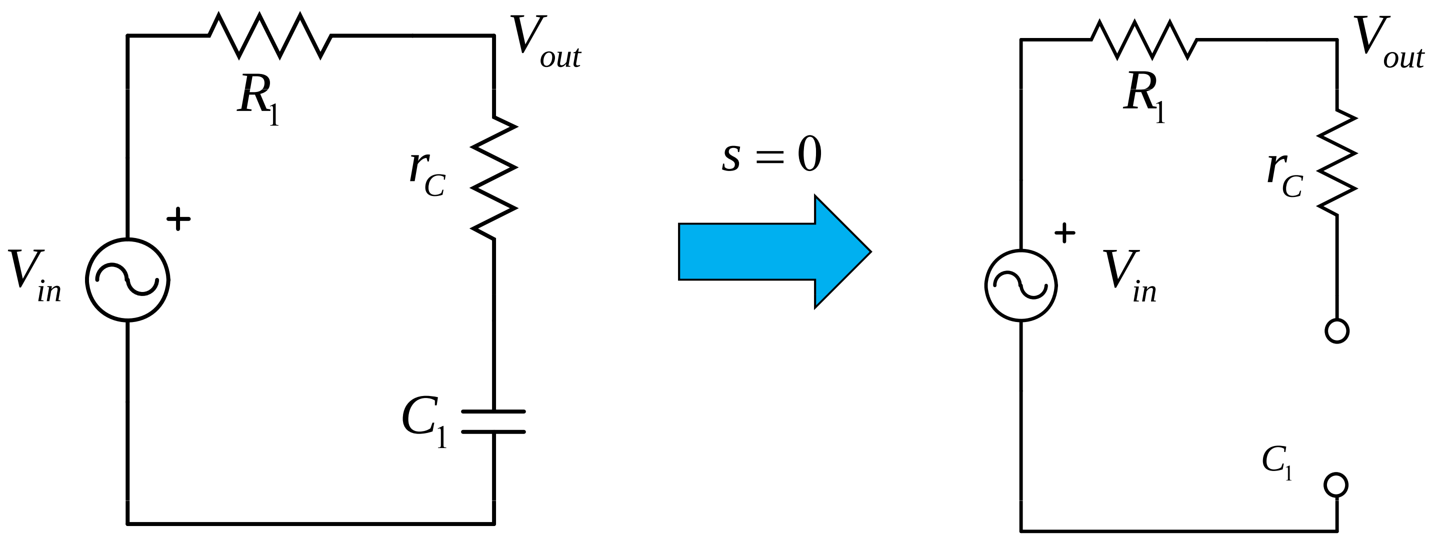
Dc state	$Z_L = 0$	Inductor is a short circuit
HF state	$Z_L = \infty$	Inductor is an open circuit

- Change the circuit depending on s

Fast Analytical Techniques at a Glance

- Look at the circuit for $s = 0$
  - Capacitor are open circuited
  - Inductors are short circuited

SPICE operating point calculation



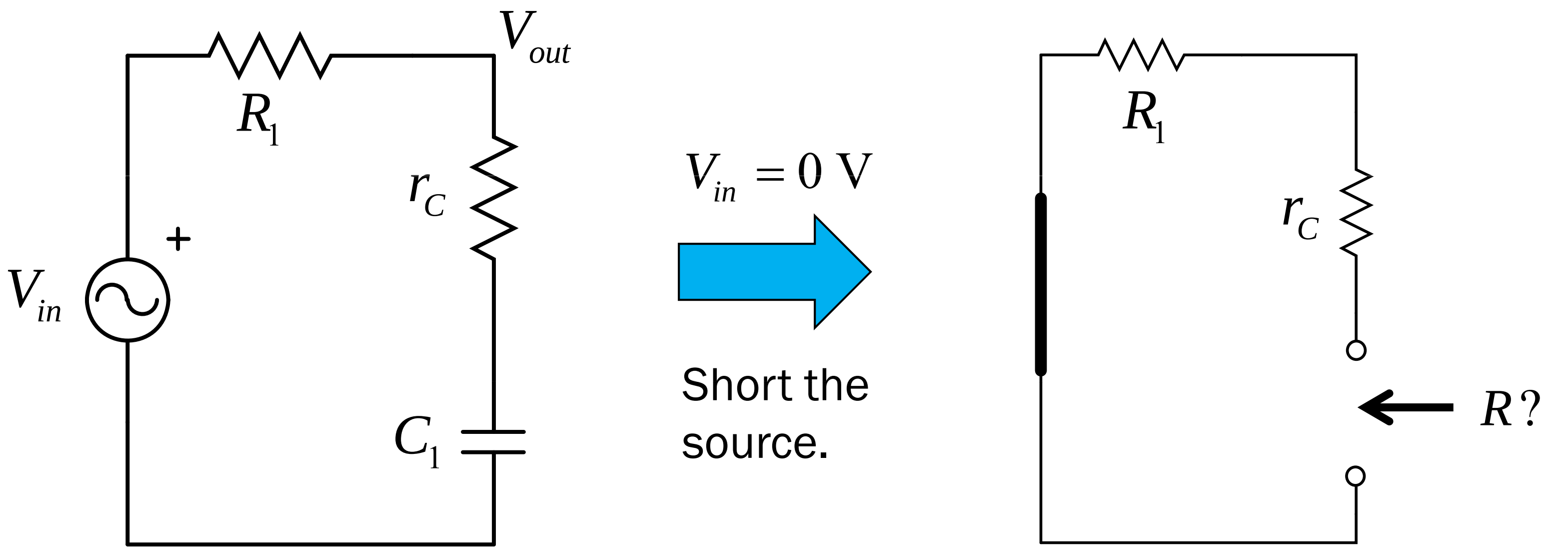
- Determine the gain in this condition

$$H_0 = V_{out} / V_{in} = 1$$

Fast Analytical Techniques at a Glance

- Look at the resistance driving the storage element

1. When the excitation is turned off, V_{in} = 0 V



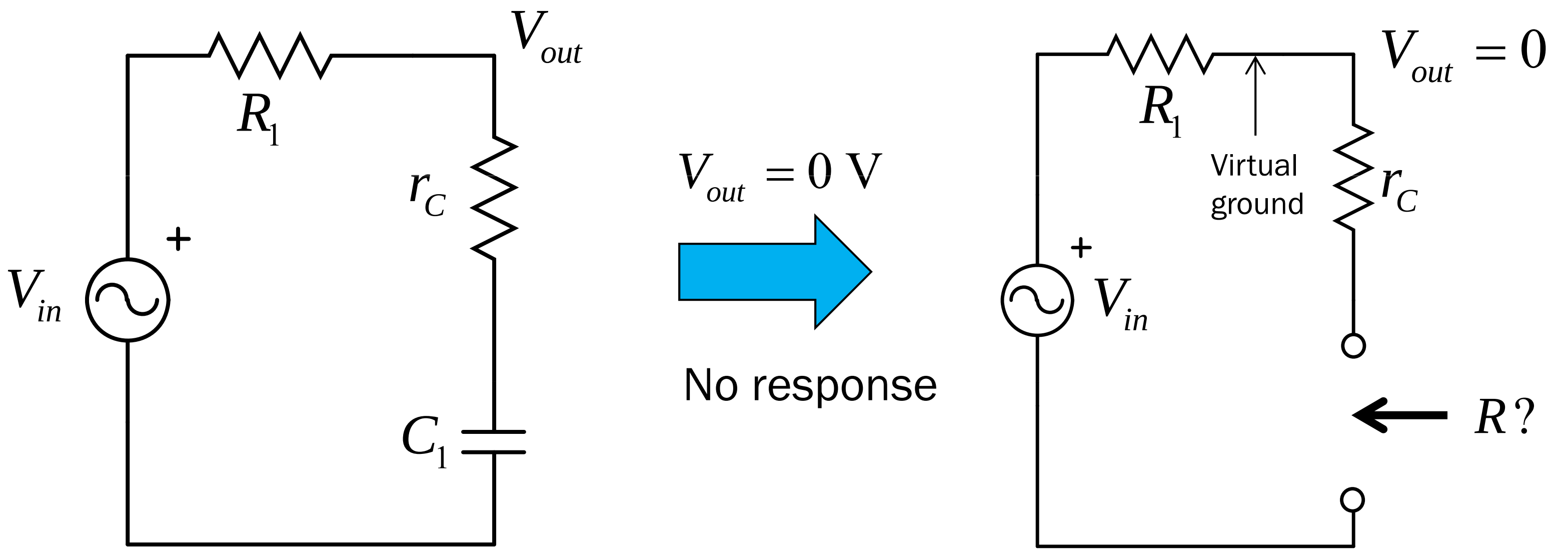
- Remove the capacitor and look into its terminals

➢ The first time constant is $\tau_1 = (r_C + R_1)C_1$

Fast Analytical Techniques at a Glance

- Look at the resistance driving the storage element

1. When the excitation is back but $V_{out} = 0$ V



- Remove the capacitor and look into its terminals

➢ The second time constant is $\tau_2 = r_C C_1$

Combining Time Constants

- By combining times constants, we have

$$H(s) = H_0 \frac{1+s\tau_2}{1+s\tau_1} = \frac{1+sr_C C_1}{1+s\left(r_C + R_1\right)C_1}$$

- Rearrange the equation to unveil a pole and a zero

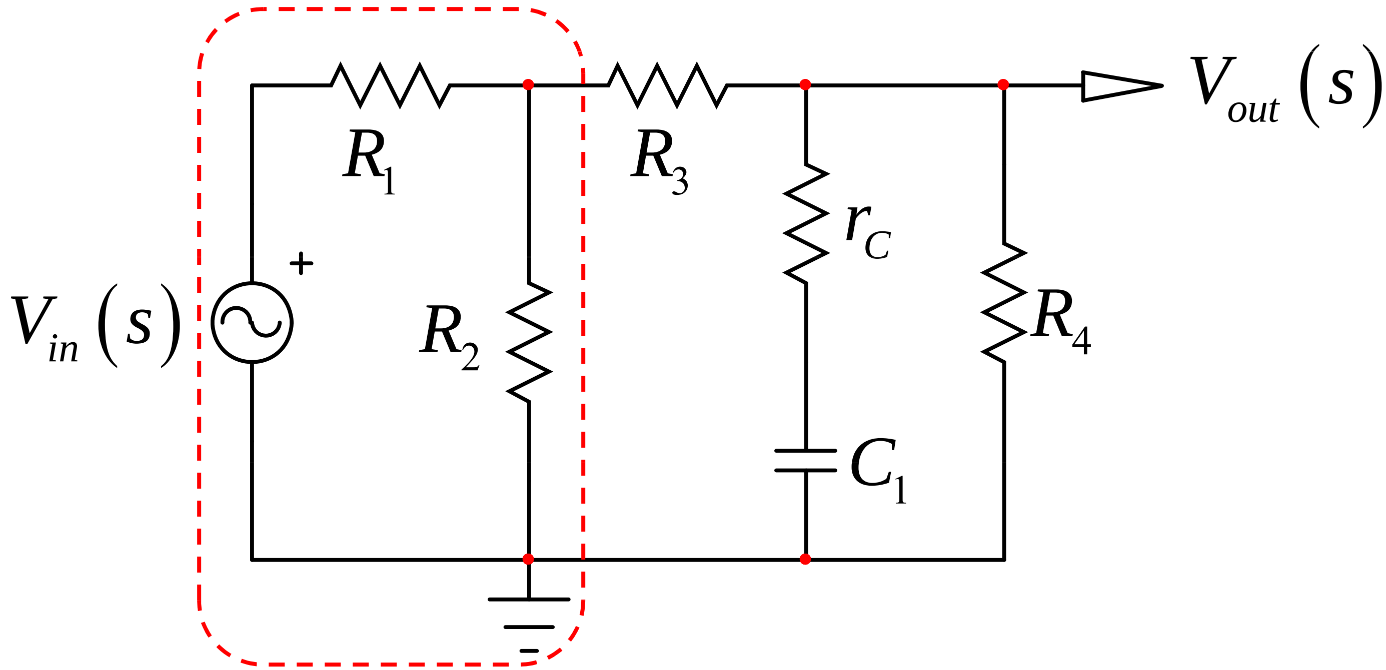
$$H(s) = H_0 \frac{1+\frac{s}{\omega_z}}{1+\frac{s}{\omega_p}} \quad \left. \right\} \quad \omega_z = \frac{1}{r_C C_1} \qquad H_0 = 1$$

$$\omega_p = \frac{1}{\left(r_C + R_1\right)C_1}$$

- This is a *low-entropy* expression

Another Example

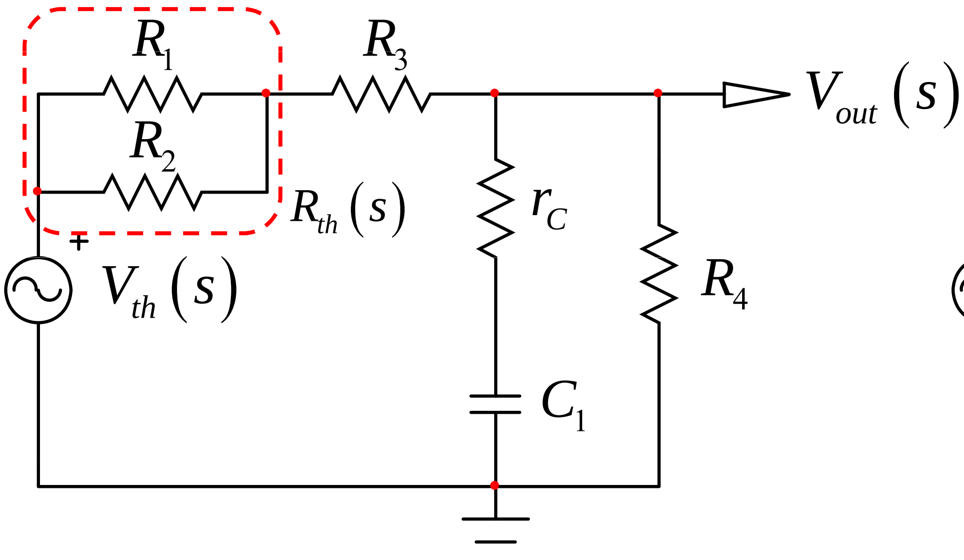
- How would you calculate V_{out} / V_{in}?

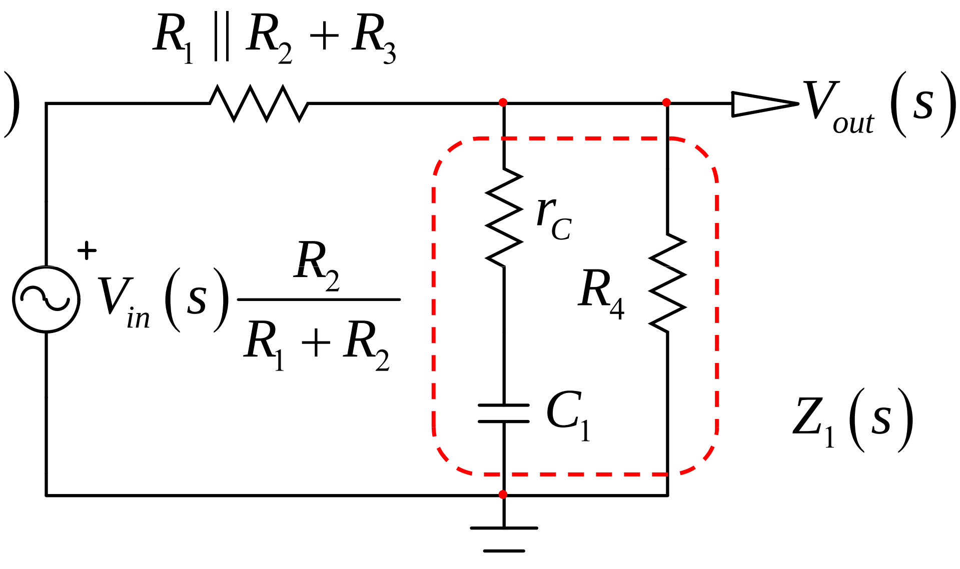


1. Transform the circuit with a Thévenin generator
2. Apply impedance divider involving C_1

Apply Impedance Divider

- Reduce circuit complexity with Thévenin





- Apply impedance divider involving Z_1 and R_{th}

$$R_{th}(s) = R_1 \| R_2 + R_3$$

$$Z_1(s) = R_4 \| \left(r_C + \frac{1}{sC_1} \right)$$

$$H(s) = \frac{Z_1(s)}{Z_1(s) + R_{th}(s)} \frac{R_2}{R_1 + R_2}$$

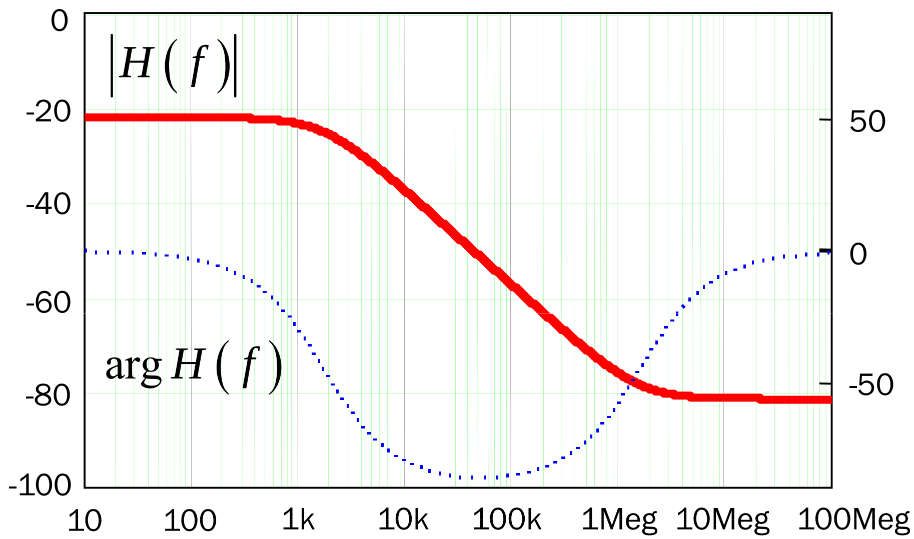
"Who you gonna call?"

High-Entropy Expression

- How do you make use of this result?

$$H_2(s) := \frac{R_2 \cdot R_4 \cdot (C_1 \cdot r_C \cdot s + 1)}{R_1 \cdot R_2 + R_1 \cdot R_3 + R_1 \cdot R_4 + R_2 \cdot R_3 + R_2 \cdot R_4 + C_1 \cdot R_1 \cdot R_2 \cdot R_4 \cdot s + C_1 \cdot R_1 \cdot R_3 \cdot R_4 \cdot s + C_1 \cdot R_2 \cdot R_3 \cdot R_4 \cdot s + C_1 \cdot R_1 \cdot R_2 \cdot r_C \cdot s + C_1 \cdot R_1 \cdot R_3 \cdot r_C \cdot s + C_1 \cdot R_1 \cdot R_4 \cdot r_C \cdot s + C_1 \cdot R_2 \cdot R_3 \cdot r_C \cdot s + C_1 \cdot R_2 \cdot R_4 \cdot r_C \cdot s}$$

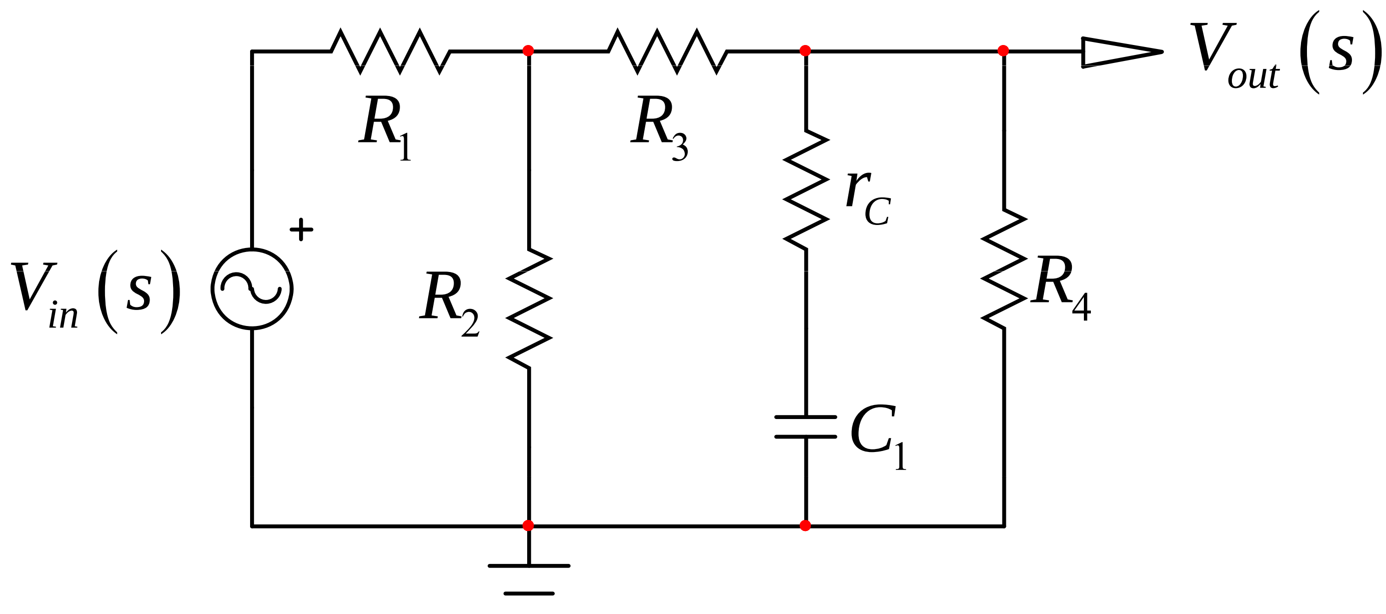
- what is the pole/zero position?
- what affects the quasi-static gain for s = 0?



You can plot the ac response but it yields no insight on what drives poles and zeros!

Applying FACTs Now

- What is the gain when V_{in} is a dc voltage?



- The capacitor is open circuited, read the schematic!

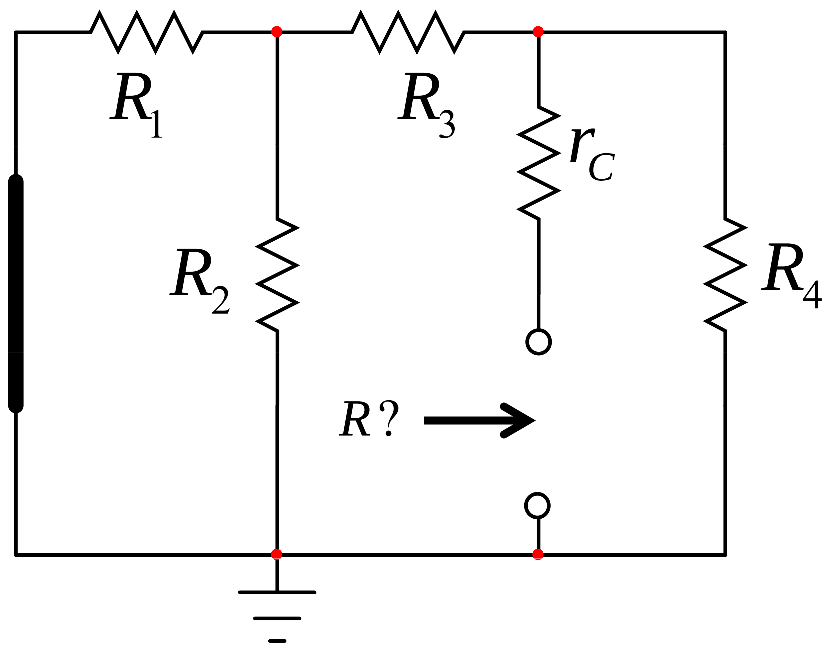
$$H_0 = \frac{R_2}{R_1 + R_2} \frac{R_4}{R_1 \| R_2 + R_3 + R_4}$$

Fast Analytical Circuits Techniques – FACTs, V. Vorpérian

Determine the First Time Constant

- Look at the resistance driving the storage element

1. When the excitation is turned off, V_{in} = 0 V

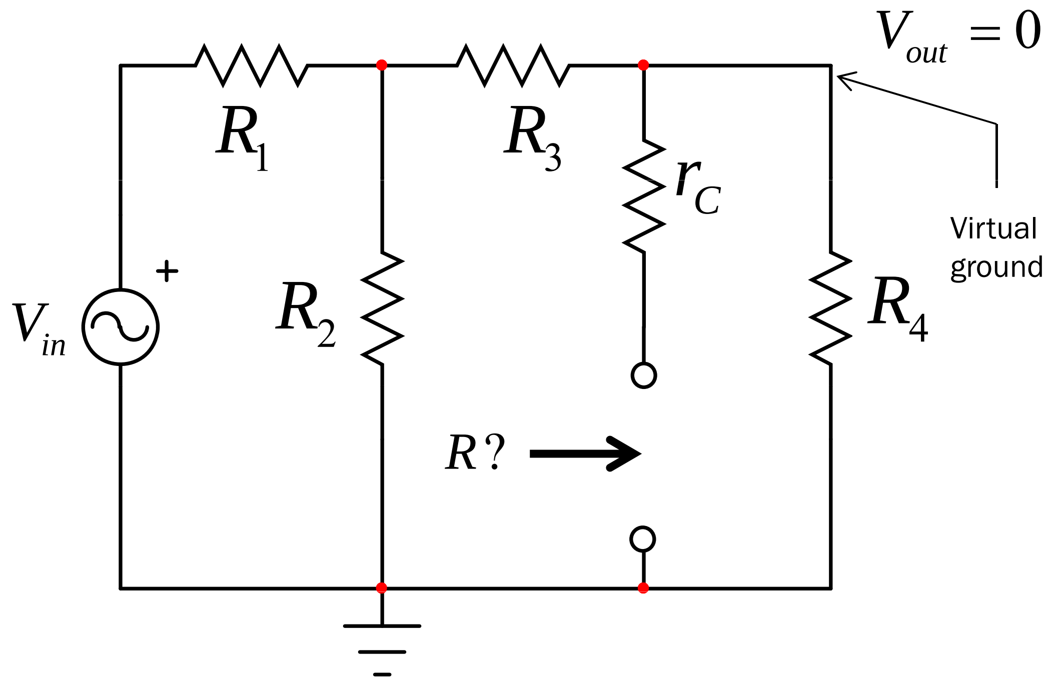


$$\tau_1 = \left[r_C + \left(R_1 \,||\, R_2 + R_3 \right) || \, R_4 \right] C_1$$

Determine the Second Time Constant

- Look at the resistance driving the storage element

1. When the excitation is back but V_{out} = 0 V



$$\tau_2 = r_C C_1$$

Assemble the Terms

- You immediately have a *low-entropy* form

$$H(s)=H_0\frac{1+\dfrac{s}{\omega_z}}{1+\dfrac{s}{\omega_p}}$$

$$H_0=\frac{R_2}{R_1+R_2}\frac{R_4}{R_1\,||\,R_2+R_3+R_4}$$

$$\omega_p=\frac{1}{\left[r_C+\left(R_1\,||\,R_2+R_3\right)||\,R_4\right]C_1}$$

$$\omega_z=\frac{1}{r_C C_1}$$

Way cool!

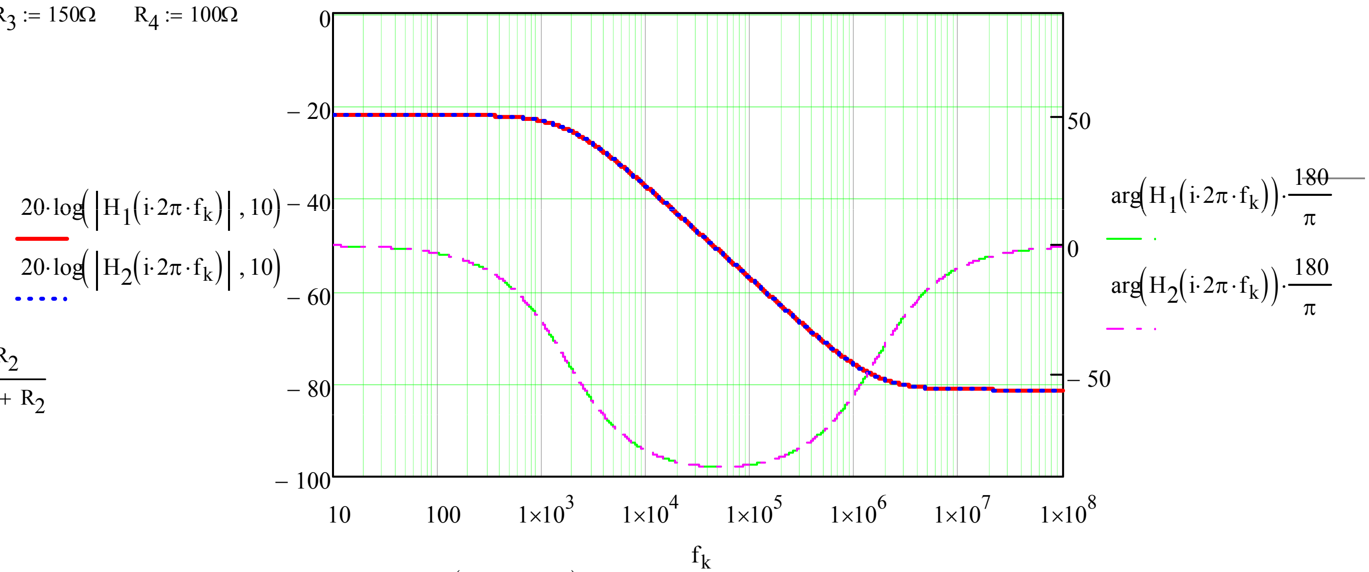
- We did not write a single line of algebra!

Use Mathcad® to Check Results

$R_1 := 1k\Omega \quad R_2 := 22k\Omega \quad r_C := 0.1\Omega \quad R_3 := 150\Omega \quad R_4 := 100\Omega$

$\|(x,y) := \frac{x \cdot y}{x + y} \quad C_1 := 1\mu F$

$$H(s) = \frac{\dfrac{R_4 \cdot \left(r_C + \frac{1}{s \cdot C_1}\right)}{R_4 + r_C + \frac{1}{s \cdot C_1}}}{\dfrac{R_4 \cdot \left(r_C + \frac{1}{s \cdot C_1}\right)}{R_4 + r_C + \frac{1}{s \cdot C_1}} + \dfrac{R_1 \cdot R_2}{R_1 + R_2} + R_3} \cdot \frac{R_2}{R_1 + R_2}$$



$$H_2(s) := \frac{R_2 \cdot R_4 \cdot (C_1 \cdot r_C \cdot s + 1)}{R_1 \cdot R_2 + R_1 \cdot R_3 + R_1 \cdot R_4 + R_2 \cdot R_3 + R_2 \cdot R_4 + C_1 \cdot R_1 \cdot R_2 \cdot R_4 \cdot s + C_1 \cdot R_1 \cdot R_3 \cdot R_4 \cdot s + C_1 \cdot R_2 \cdot R_3 \cdot R_4 \cdot s + C_1 \cdot R_1 \cdot R_2 \cdot r_C \cdot s + C_1 \cdot R_1 \cdot R_3 \cdot r_C \cdot s + C_1 \cdot R_1 \cdot R_4 \cdot r_C \cdot s + C_1 \cdot R_2 \cdot R_3 \cdot r_C \cdot s + C_1 \cdot R_2 \cdot R_4 \cdot r_C \cdot s}$$

$\tau_2 := C_1 \cdot \left[r_C + \left(R_1 \| R_2 + R_3\right) \| R_4\right] = 91.812 \cdot \mu s$

$\tau_1 := C_1 \cdot r_C = 100 \cdot ns$

$H_0 := \frac{R_4}{R_4 + R_1 \| R_2 + R_3} \cdot \frac{R_2}{R_1 + R_2} = 0.079$

$H_1(s) := H_0 \cdot \frac{1 + s \cdot \tau_1}{1 + s \cdot \tau_2}$

Superimposing both transfer functions, matching should be perfect. If not, there is mistake.

Public Information
Christophe Basso –APEC 2016
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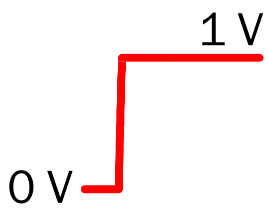
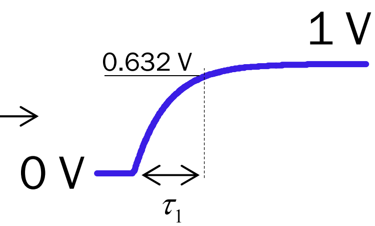
Time Constants

- Response to a step input is described by a time constant

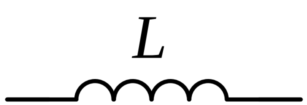


$$\frac{1}{1+s\tau_1}$$

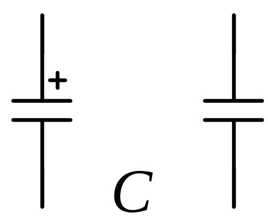


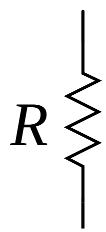
1st-order linear system

- A time constant “tau” is associated with a reactance



…and a resistance R





$$\tau = RC\,[\mathrm{s}]$$

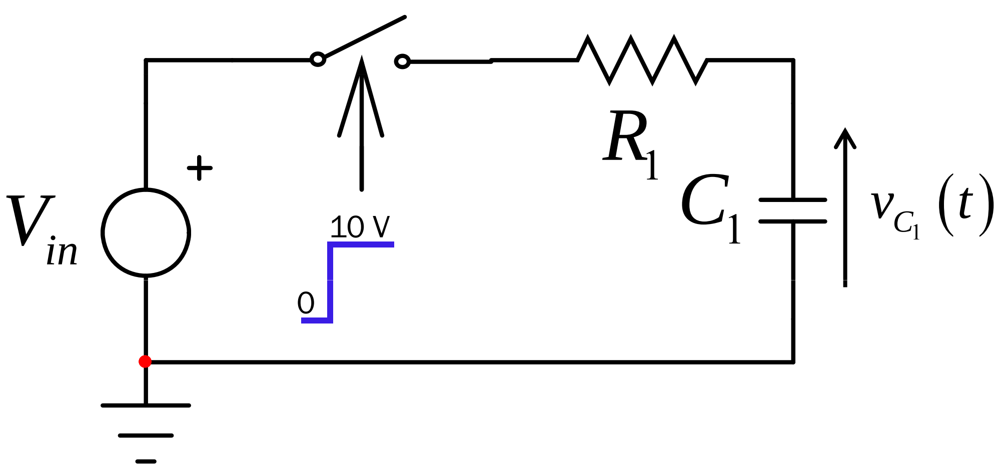
$$\tau = \frac{L}{R}\,[\mathrm{s}]$$

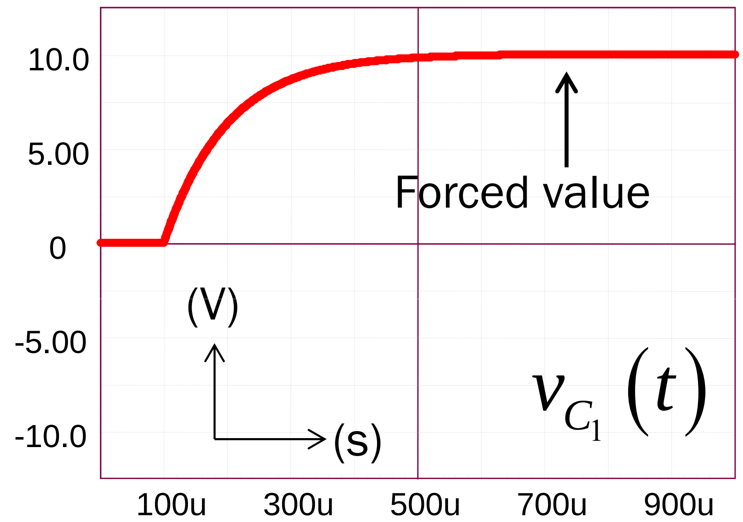
Time-Domain Response

- The time-domain response $y(t)$ of a linear system is

$$y(t) = \underbrace{r_f(t)}_{\text{Forced response}} + \underbrace{r_n(t)}_{\text{Natural response}}$$

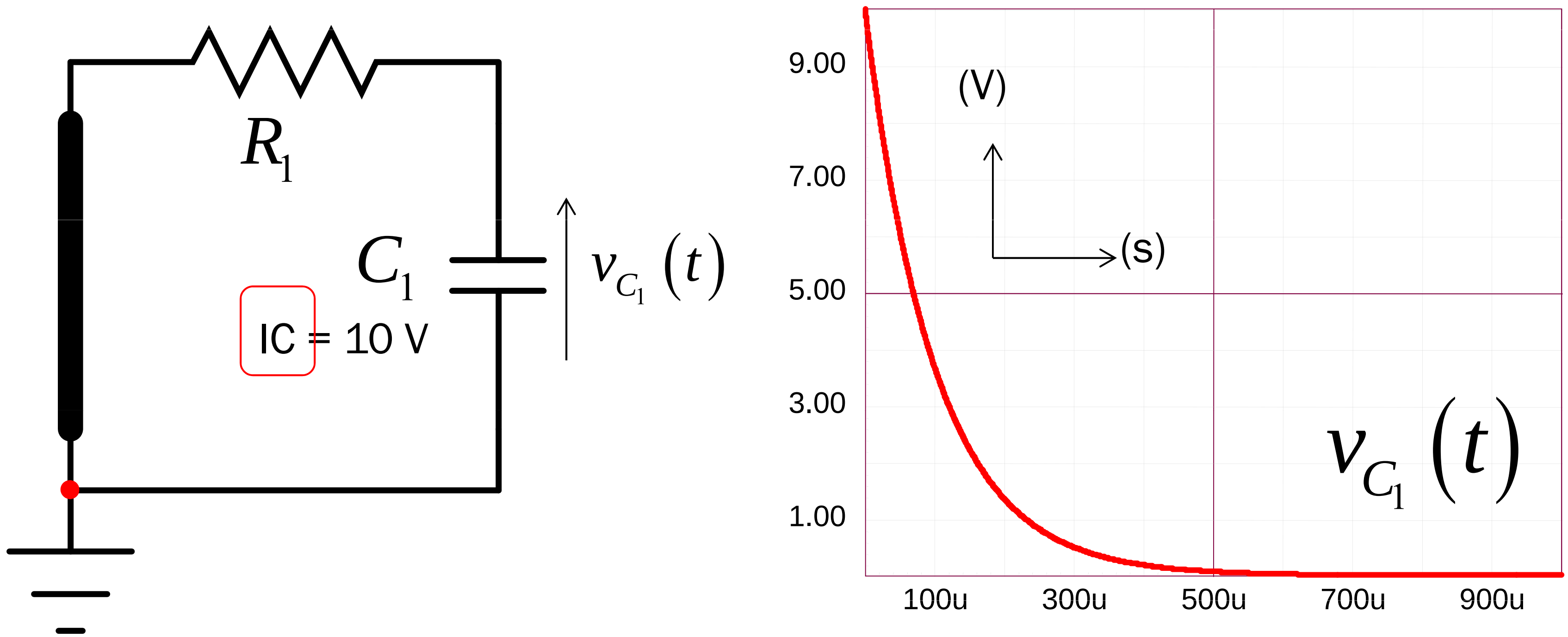
- The first term depends on the excitation – the force





Natural Response

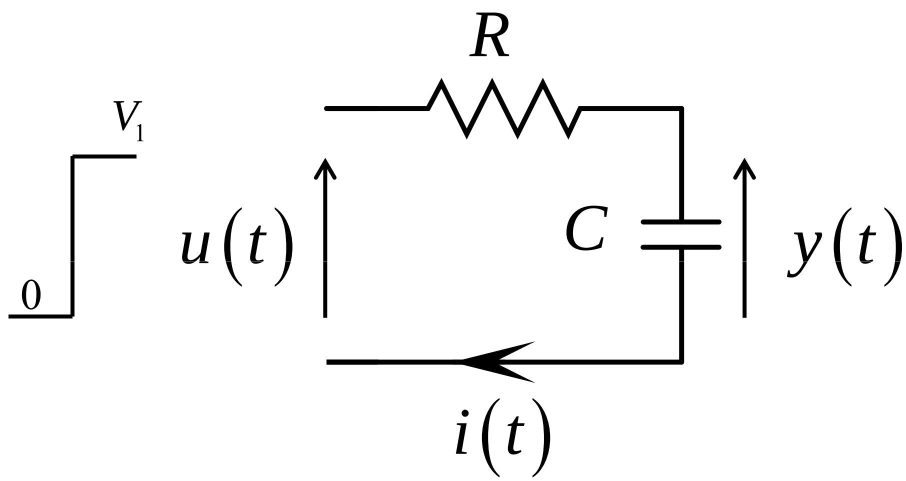
- Natural response solely involves initial conditions



➡ You don't need a source for the natural response

Time Constant Involving a Capacitor

- Assume a simple low-pass RC filter



$$u(t) = Ri(t) + y(t)$$

$$i(t) = C\frac{dv_C(t)}{dt}$$

Initial capacitor voltage is V_0

$$y(t) = u(t) - RC\frac{dv_C(t)}{dt} = u(t) - RC\frac{dy(t)}{dt}$$

- The state variable associated with C is its voltage, x_2

Time Domain to Laplace

- Take the Laplace transform of the time-domain equation

$$Y(s) = \mathcal{L}\{y(t)\} = U(s) - RC\left(sY(s) - V_0\right)$$

$$Y(s) = \frac{U(s)}{1+s\boxed{RC}} + \frac{RCV_0}{1+sRC}$$

$\tau = RC$ time constant

- Considering 0-V initial conditions, $v_C(0) = 0$

$$Y(s) = \frac{U(s)}{1+sRC} + \underbrace{\cancel{\frac{RCV_0}{1+sRC}}}_{=0} \Rightarrow \frac{Y(s)}{U(s)} = \frac{1}{1+s\boxed{\tau}}$$

1st-order transfer function

Forced and Natural Responses

- Assume that input voltage U is a step function

$$\mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{V_1}{s}\frac{1}{1+sRC}\right\} + \mathcal{L}^{-1}\left\{\frac{RCV_0}{1+sRC}\right\} \qquad \tau = RC$$

- Use inverse Laplace-transform tables to obtain

$$y(t) = \boxed{V_1\left(1-e^{-\frac{t}{\tau}}\right)} + \boxed{V_0 e^{-\frac{t}{\tau}}} \Rightarrow y(t) = r_f(t) + r_n(t)$$

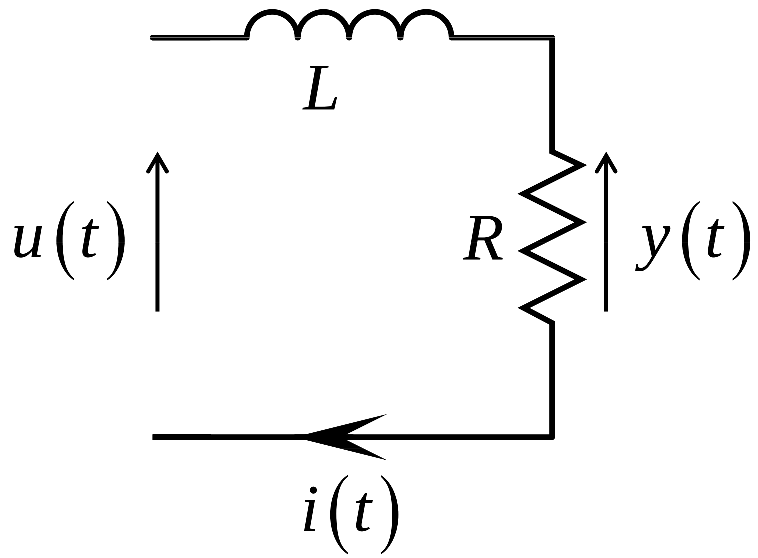
Forced response
No initial conditions

Natural response
No source contribution

Time constant

Time Constant Involving an Inductor

- Assume a simple low-pass LR filter



$$u(t) = L\frac{di(t)}{dt} + y(t)$$

$$y(t) = Ri(t)$$

Initial inductor current is I_0

$$y(t) = u(t) - L\frac{di(t)}{dt}$$

- The state variable associated with L is its current, x_1

Laplace Transform

- Take the Laplace transform of the time-domain equation

$$Y(s) = \mathcal{L}\{y(t)\} = U(s) - L\left(sI(s) - I_0\right) \qquad I(s) = \frac{Y(s)}{R}$$

$$Y(s) = U(s) - L\left(s\frac{Y(s)}{R} - I_0\right) \qquad Y(s) = \frac{U(s)}{1 + s\frac{L}{R}} + \frac{LI_0}{1 + s\boxed{\frac{L}{R}}}$$

$$\tau = \frac{L}{R}$$

Time constant

- Considering 0-A initial conditions, $i_L(0) = 0$

$$Y(s) = \frac{U(s)}{1 + s\frac{L}{R}} + \underbrace{\cancel{\frac{LI_0}{1 + s\frac{L}{R}}}}_{=0} \Rightarrow \frac{Y(s)}{U(s)} = \frac{1}{1 + s\boxed{\tau}}$$

1st-order transfer function

Response to an Input Step

- Now assume that input voltage U is a step function

$$U(s) = \frac{V_1}{s}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{V_1}{s}\frac{1}{1+s\frac{L}{R}}\right\} + \mathcal{L}^{-1}\left\{\frac{LI_o}{1+s\frac{L}{R}}\right\} \qquad \tau = \frac{L}{R}$$

$$y(t) = \boxed{V_1\left(1-e^{-\frac{t}{\tau}}\right)} + \boxed{LI_0 e^{-\frac{t}{\tau}}}$$

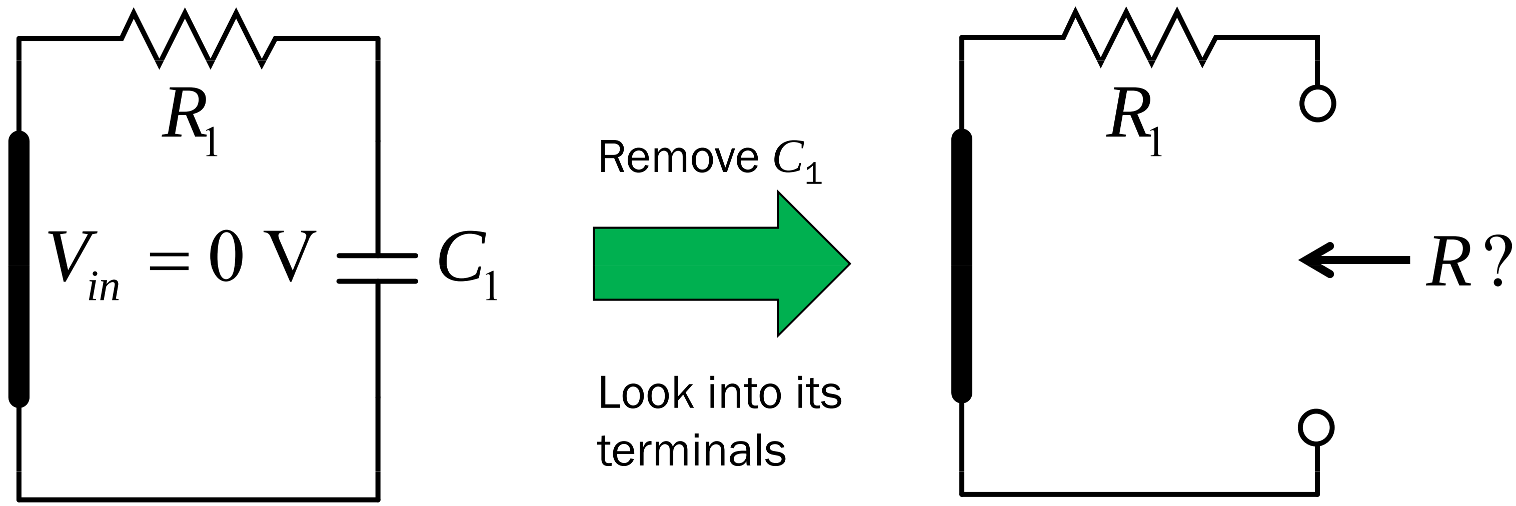
Forced response
No initial conditions

Natural response
No source contribution

Time constant

Natural Time Constant

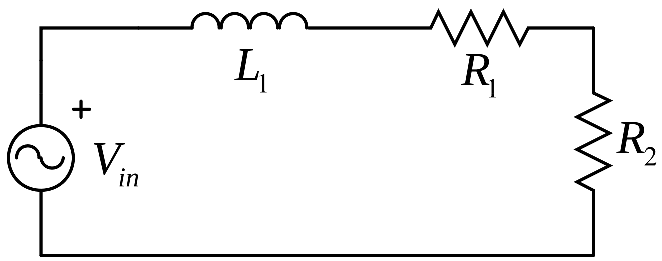
- ❑ The time constant tau plays a role in r_f and r_n
- ❑ How can we determine tau in the simplest way?
- ❖ Look at natural response circuit where V_{in} is off



- ❑ What resistance do you see? R_1 then $\tau = R_1 C_1$

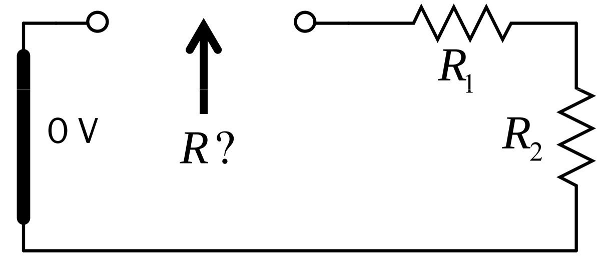
Setting the Excitation to Zero

- Turning the excitation off means
  - A 0-V source becomes a short circuit
  - A 0-A generator is an open circuit and disappears

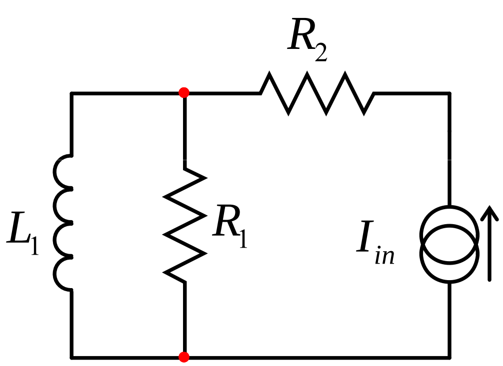


Set source

to 0 V

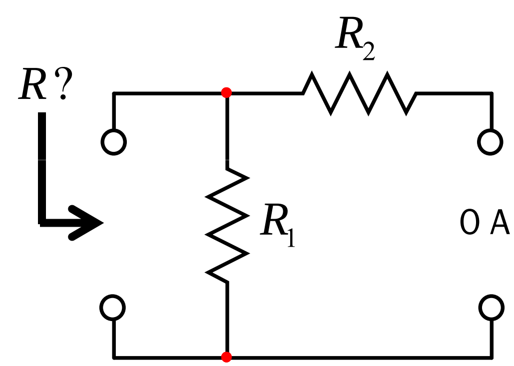


$$\tau = \frac{L_1}{R_1 + R_2}$$



Set source

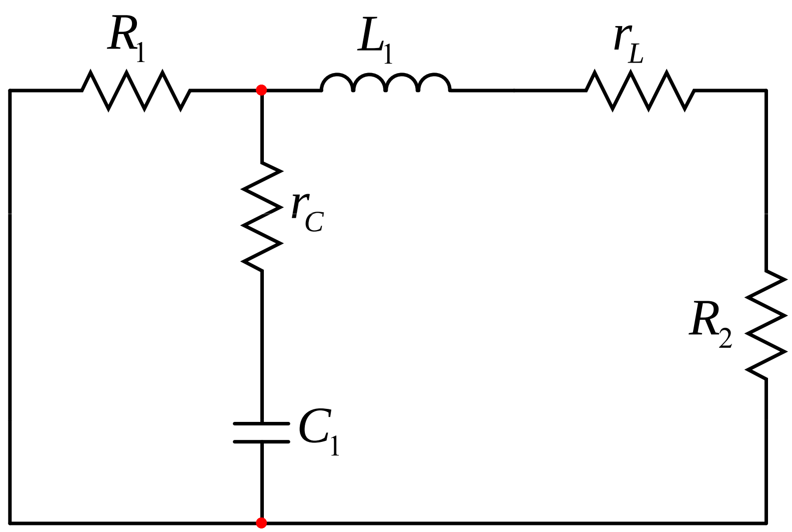
to 0 A



$$\tau = \frac{L_1}{R_1}$$

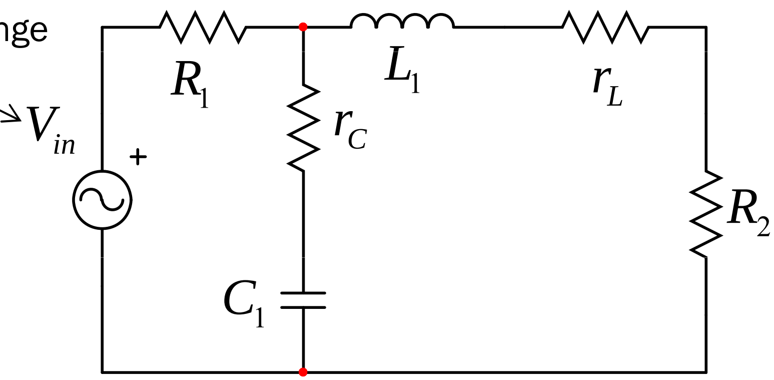
Excitation Plays no Role

- Time constants are part of the network structure

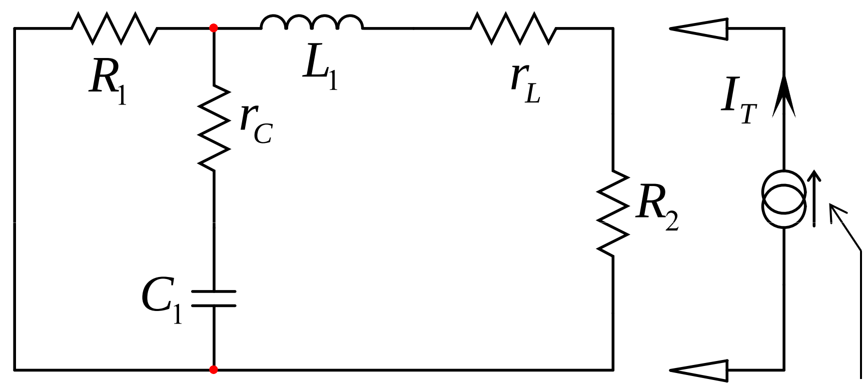


Natural network structure

Set V_{in} to 0 V:
no change



Voltage excitation



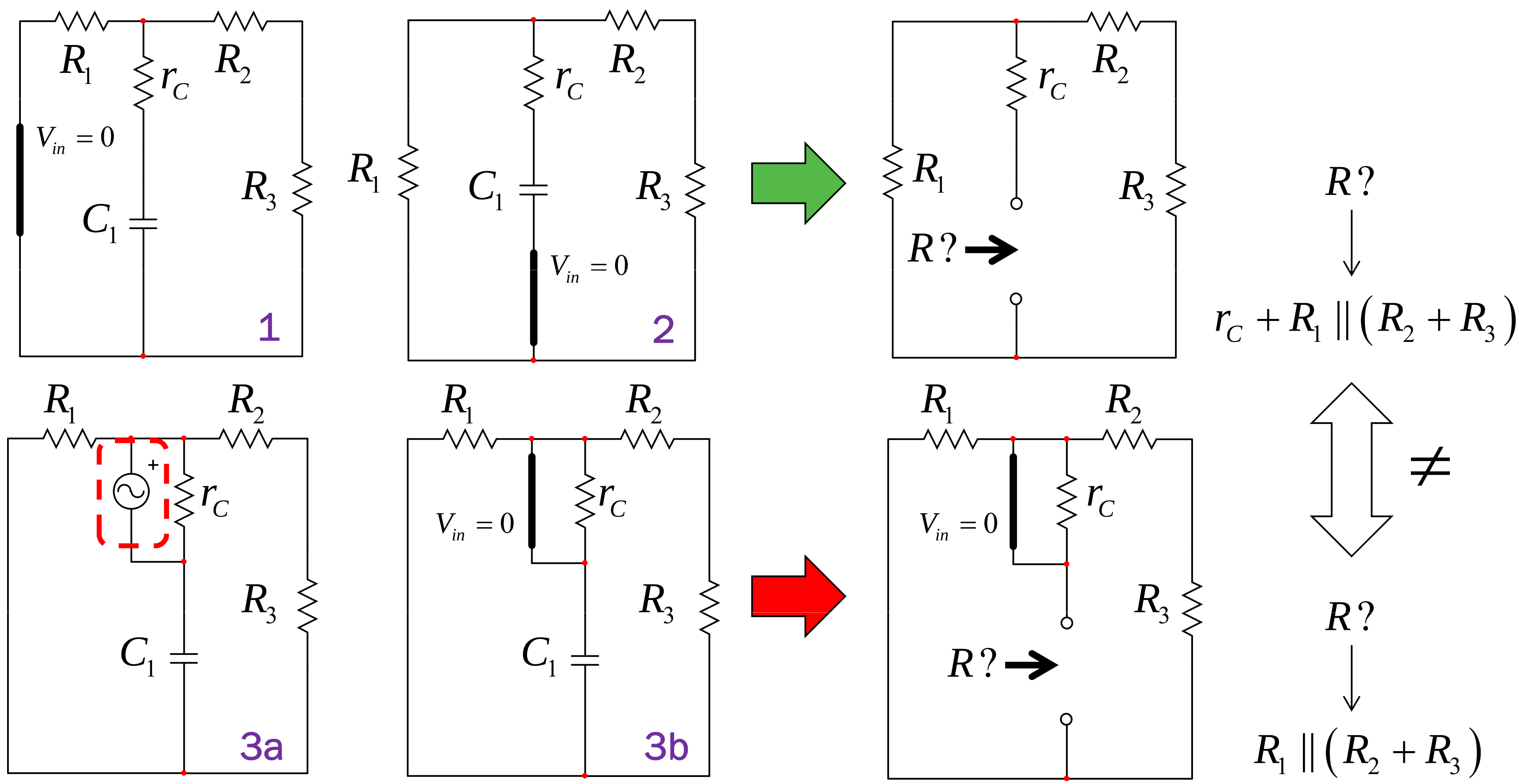
Current excitation

Set I_T to 0 A:
no change

- When excitation is off, the structure remains the same

Does Excitation Change the Structure?

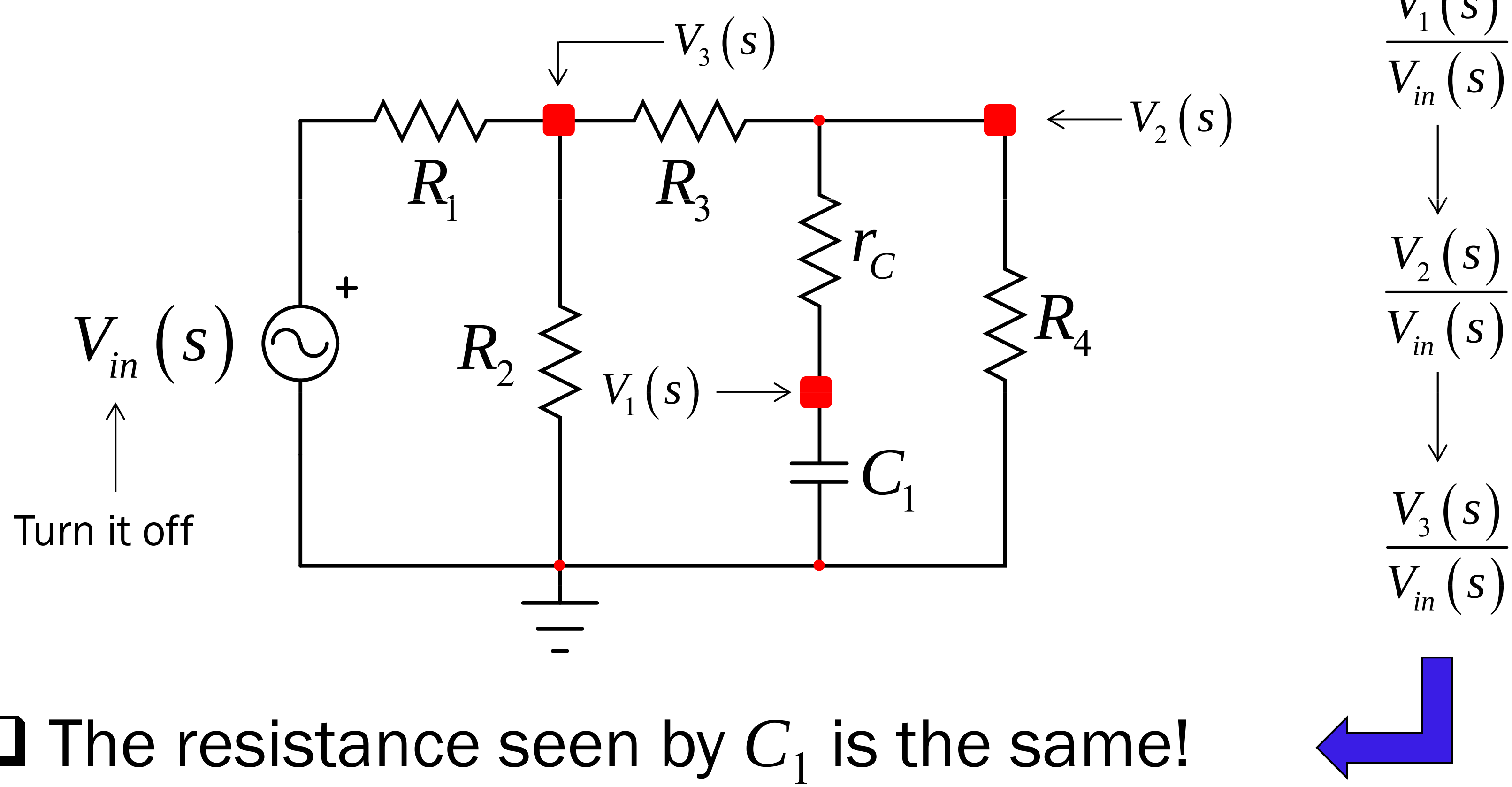
- The time constant does not change for $V_{in} = 0$ V



Modified structure!

Probing Does not Affect Time Constants

- You can observe the response at any place
  - Time constants remain the same



- The resistance seen by C_1 is the same!

Denominator and Time Constants

- The response of a SISO system is given by:

$$y(t) = r_n(t) + r_f(t) = \sum_{i=1}^{n} C_i e^{p_i t} + r_f(t)$$

C are the exponential terms coefficients
n is the system order
p are the poles of the systems

- Assume the following 3rd-order transfer function:

$$H(s) = \frac{N(s)}{D(s)} = \frac{\left[(s+2)^2 + 4\right]}{\left[(s+1)^2 + 1\right]} \frac{s-1}{s+3} \qquad H(0) = \frac{\left[(2)^2 + 4\right]}{\left[(1)^2 + 1\right]} \frac{(-1)}{3} = -\frac{4}{3}$$

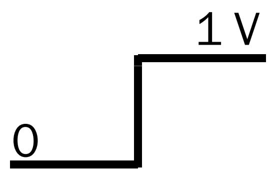
3rd order denominator, 3 poles: $p_3 = -3$ $\quad p_{1,2} = \pm j - 1$

SISO: single-output single-input

SISO Response to a Step Input

- Multiply the transfer function by a step input

$$Y(s) = \frac{1}{s} H(s) \longrightarrow$$



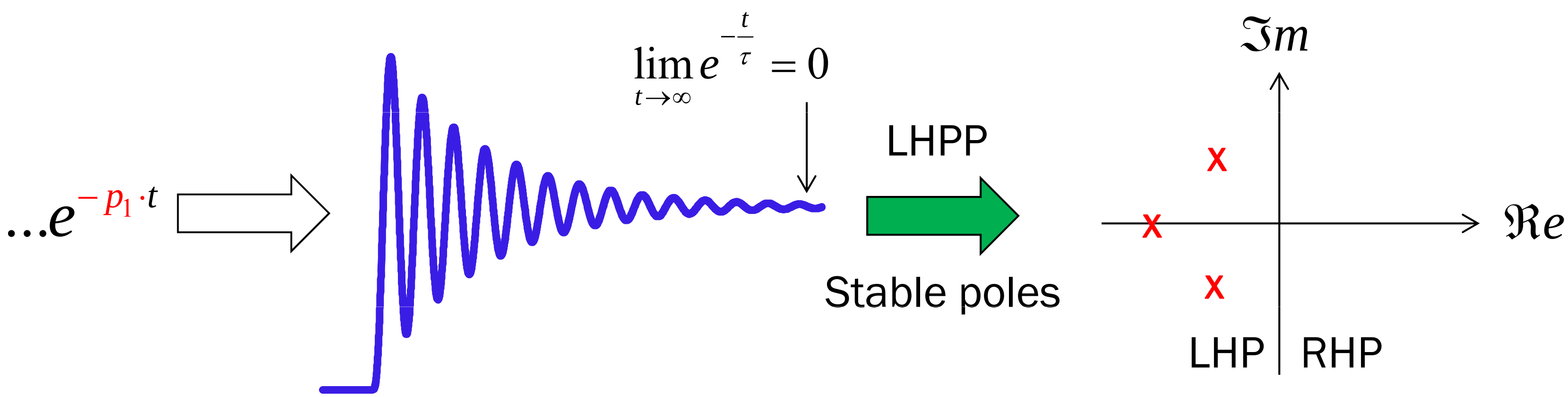
- Extract the time-domain response

$$\mathcal{L}_s^{-1}\{Y(s)\} = y(t) = \underbrace{\frac{4}{3} e^{\overset{\text{roots}}{\downarrow}{-3}t} + \cos(t) e^{\overset{\text{roots}}{\downarrow}{-1}t} + 3\sin(t) e^{\overset{\text{roots}}{\downarrow}{-1}t}}_{r_n(t)} \underbrace{- \frac{4}{3}}_{r_f(t)}$$

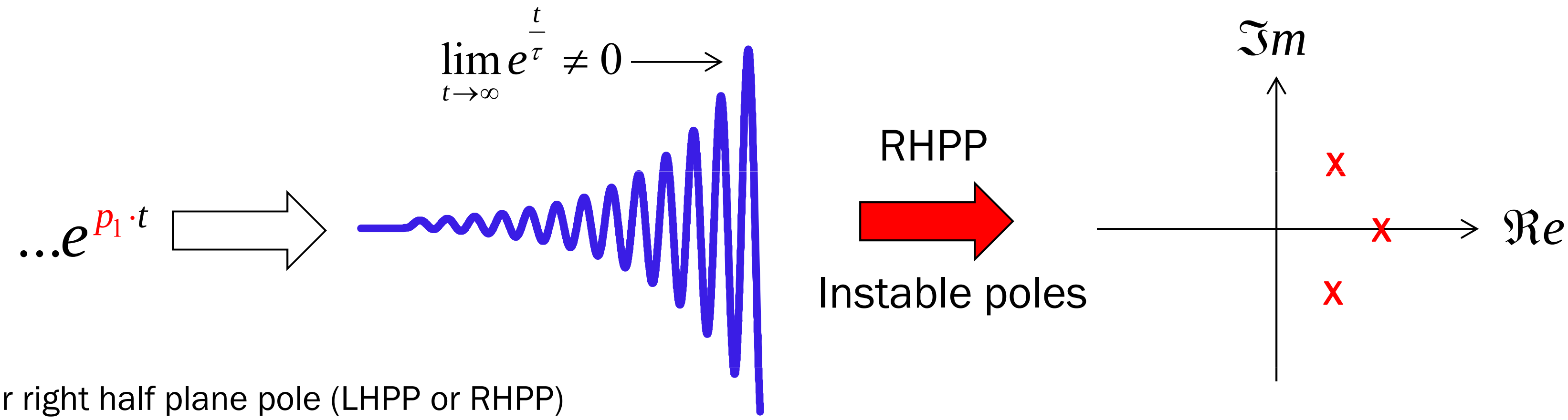
- Poles appear in the exponential power terms

Poles and Natural Time Constants

- A negative sign implies a decaying term



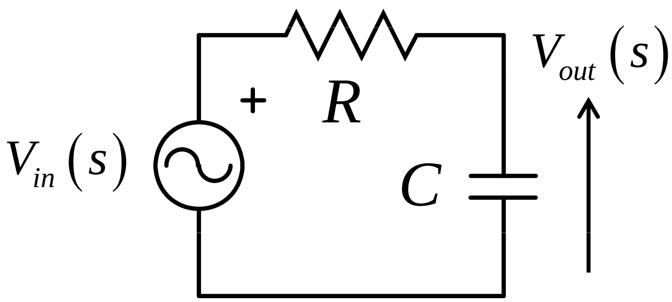
- A positive sign means it is an increasing term



Left or right half plane pole (LHPP or RHPP)

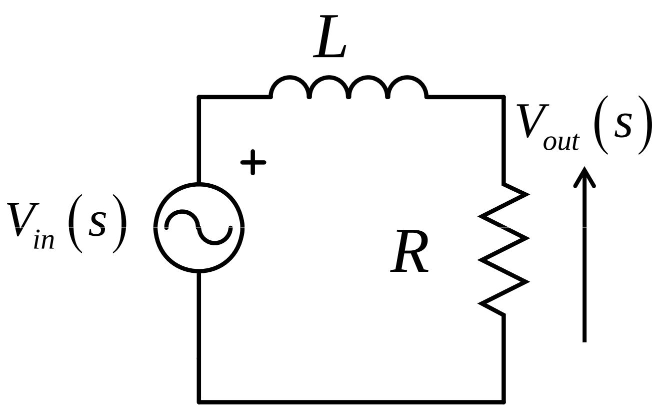
Time Constant and Pole – 1^{st} Order

- In 1^{st}-order systems, a pole is the inverse of the time constant



$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1+sRC} = \frac{1}{1+s\tau} = \frac{1}{1+\dfrac{s}{\omega_p}}$$

$$\omega_p = \frac{1}{RC} = \frac{1}{\tau}$$

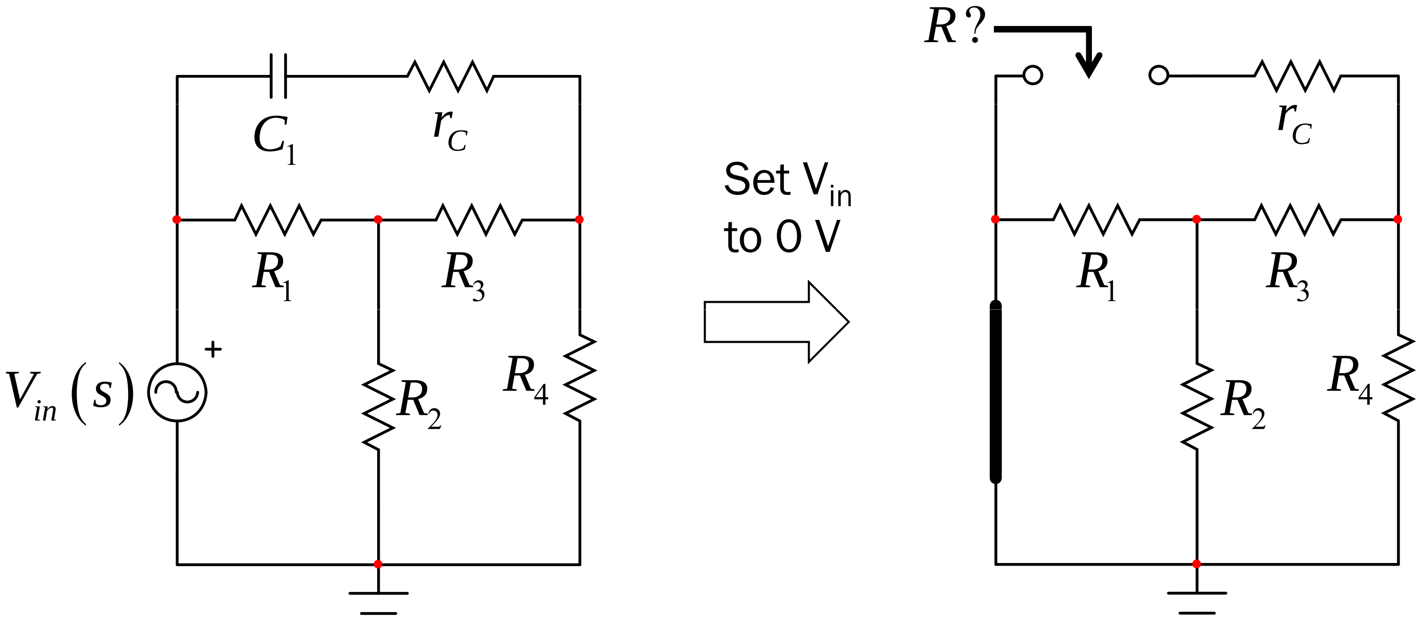


$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1+s\dfrac{L}{R}} = \frac{1}{1+s\tau} = \frac{1}{1+\dfrac{s}{\omega_p}}$$

$$\omega_p = \frac{R}{L} = \frac{1}{\tau}$$

Determining the Time Constant

- Find the time constant to obtain the pole



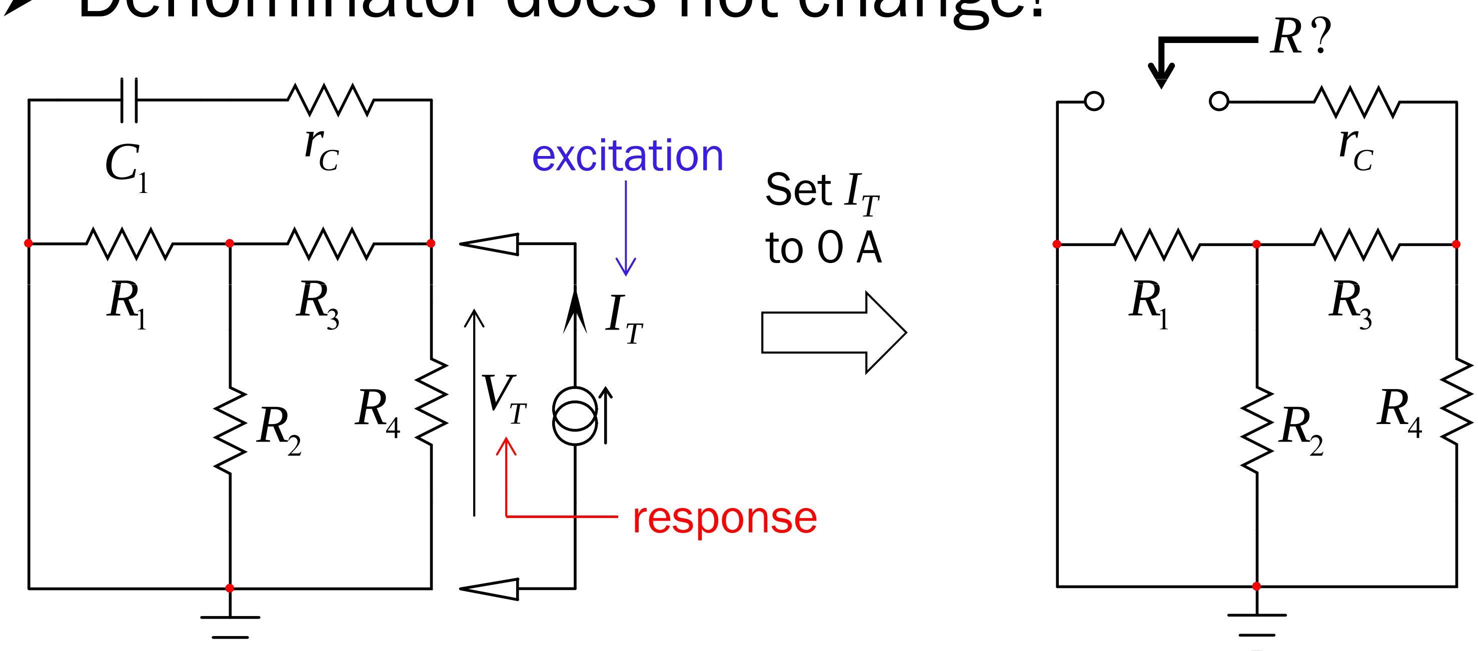
$$\tau = C_1\left[r_C + \left(R_1 \parallel R_2 + R_3\right) \parallel R_4\right]$$

$$D(s) = 1 + s\tau = 1 + \frac{s}{\omega_p}$$

$$\omega_p = \frac{1}{C_1\left[r_C + \left(R_1 \parallel R_2 + R_3\right) \parallel R_4\right]}$$

Same Denominator for Z_{out}

- A current generator does not alter the structure
  - Denominator does not change!

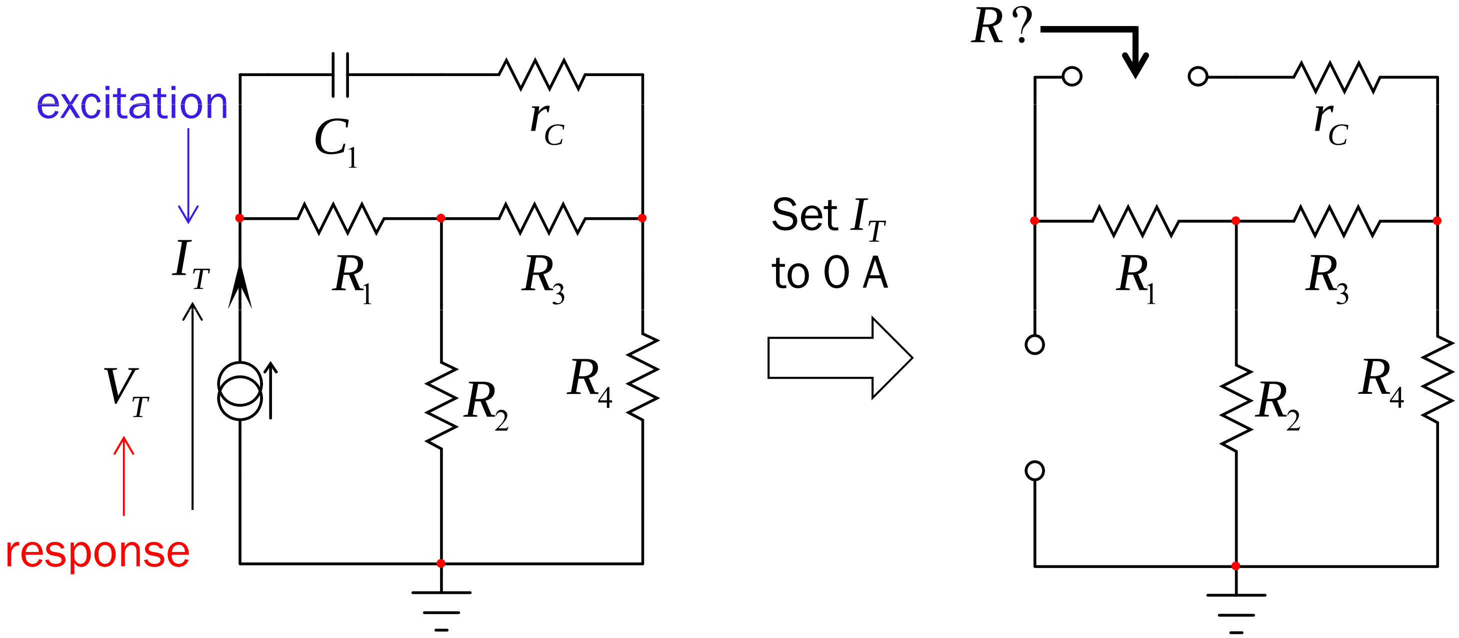


- Transfer function keeps the same denominator

$$D(s) = 1 + sC_1\left[r_C + \left(R_1 \parallel R_2 + R_3\right) \parallel R_4\right]$$

Denominator Changes for Z_{in}

- Series insertion of current source alters the structure

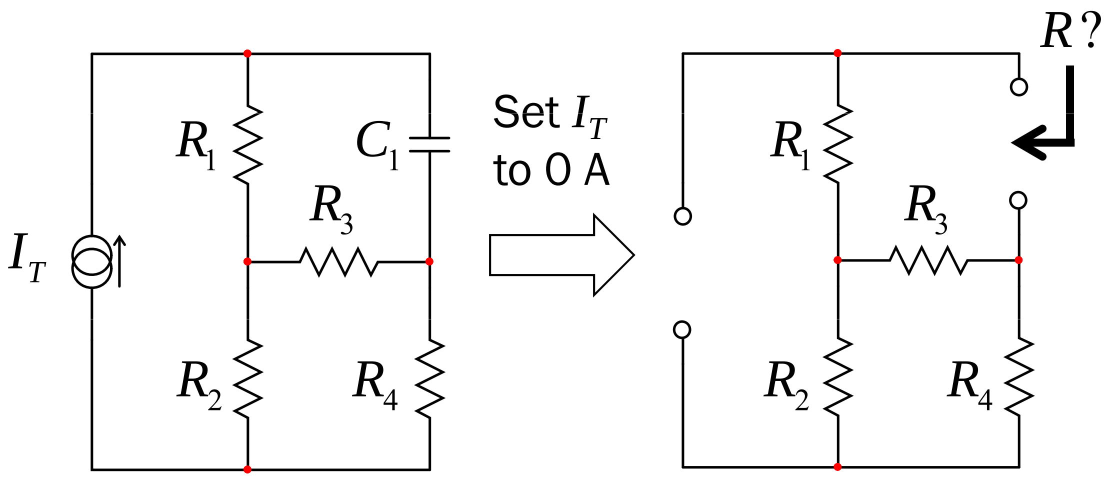


- Time constant is changed, cannot reuse $D(s)$

$$\tau = C_1\left[r_C + R_1 + R_3 \,||\, \left(R_2 + R_4\right)\right] \quad \omega_p = \frac{1}{C_1\left[r_C + R_1 + R_3 \,||\, \left(R_2 + R_4\right)\right]}$$

Find the Time Constants

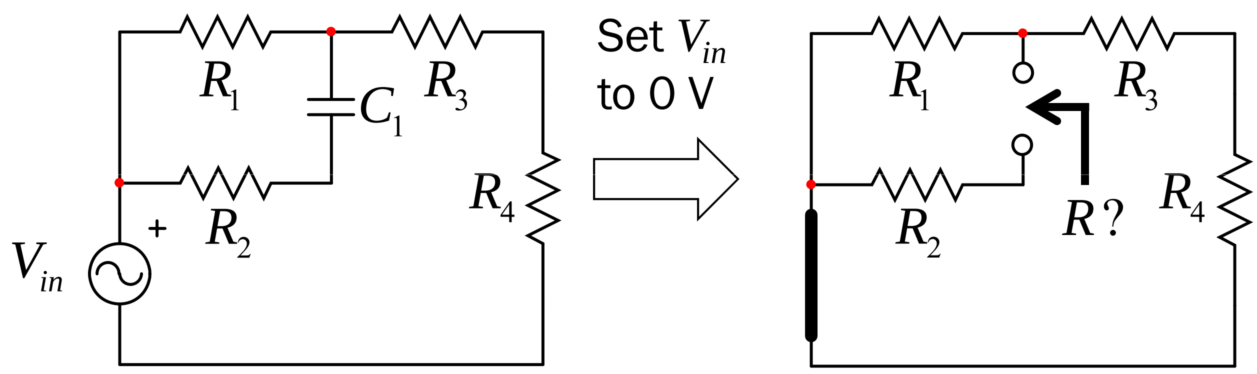
- Find the time constants when excitation is set to 0



$$\tau = C_1\left[R_1 + R_3 \| \left(R_2 + R_4\right)\right]$$

$$\downarrow$$

$$\omega_p = \frac{1}{C_1\left[R_1 + R_3 \| \left(R_2 + R_4\right)\right]}$$



$$\tau = C_1\left[R_2 + R_1 \| \left(R_3 + R_4\right)\right]$$

$$\downarrow$$

$$\omega_p = \frac{1}{C_1\left[R_2 + R_1 \| \left(R_3 + R_4\right)\right]}$$

Course Agenda

- ❑ What is a Transfer Function?
- ❑ Why do We Need New Analytical Techniques?
- ❑ Time Constants and Poles
- ❑ Identifying the Zeros
- ❑ The Null Double Injection
- ❑ 2nd-Order Networks
- ❑ The PWM Switch Model
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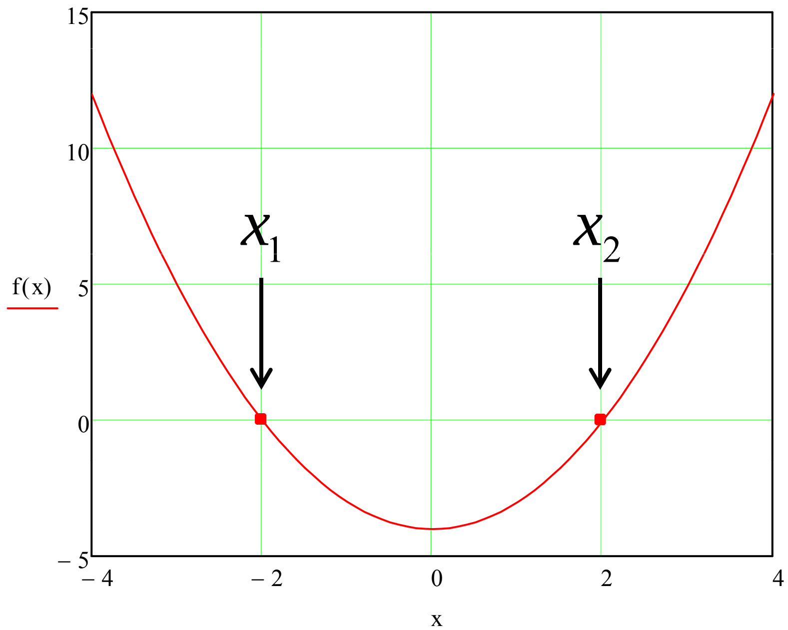
- ❑ A CCM Buck-Boost in Voltage Mode

Zero: the Mathematical Definition

- A zero is the root of the equation $f(x)=0$



$$f(x)=x^2-4$$

$$f(x)=0$$

$$\downarrow$$

$$x_1=-2$$

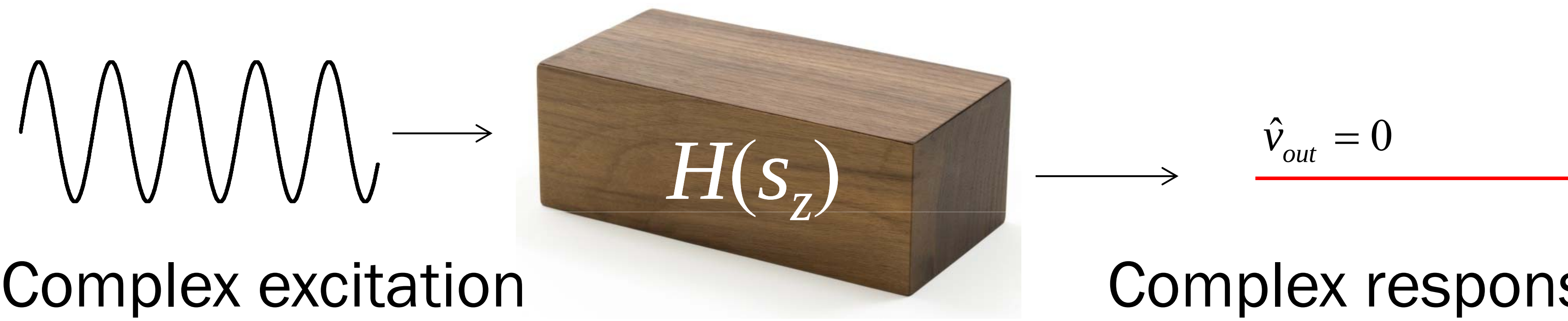
$$x_2=2$$

- Transfer function zeros are the numerator roots

$$N(s)=0 \longrightarrow s_{z_1}, s_{z_2} \ldots$$

Nulling the Response

- If the numerator is 0, then the response is also 0



Complex excitation

$$s = s_z$$

Complex response

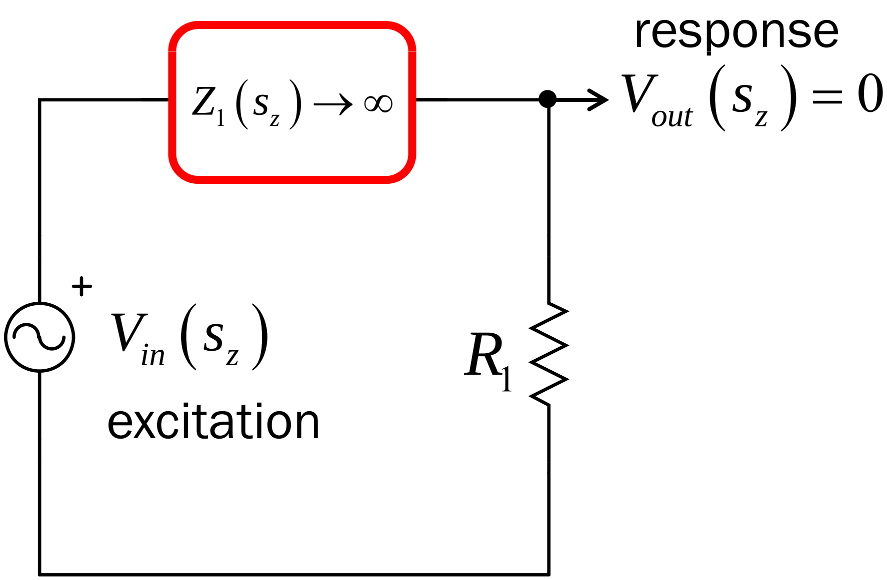
$$N\left(s_z\right) = 0$$

- What is happening in the box when $s = s_z$?

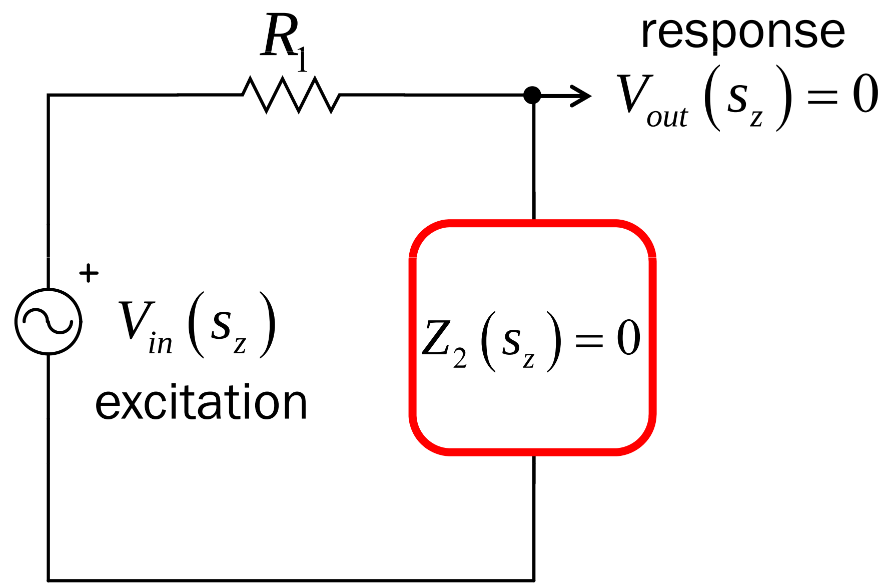
➡ The excitation does not generate a response

How Does the Response Disappear?

- The signal is lost in the *transformed* network



A series impedance
becomes infinite.

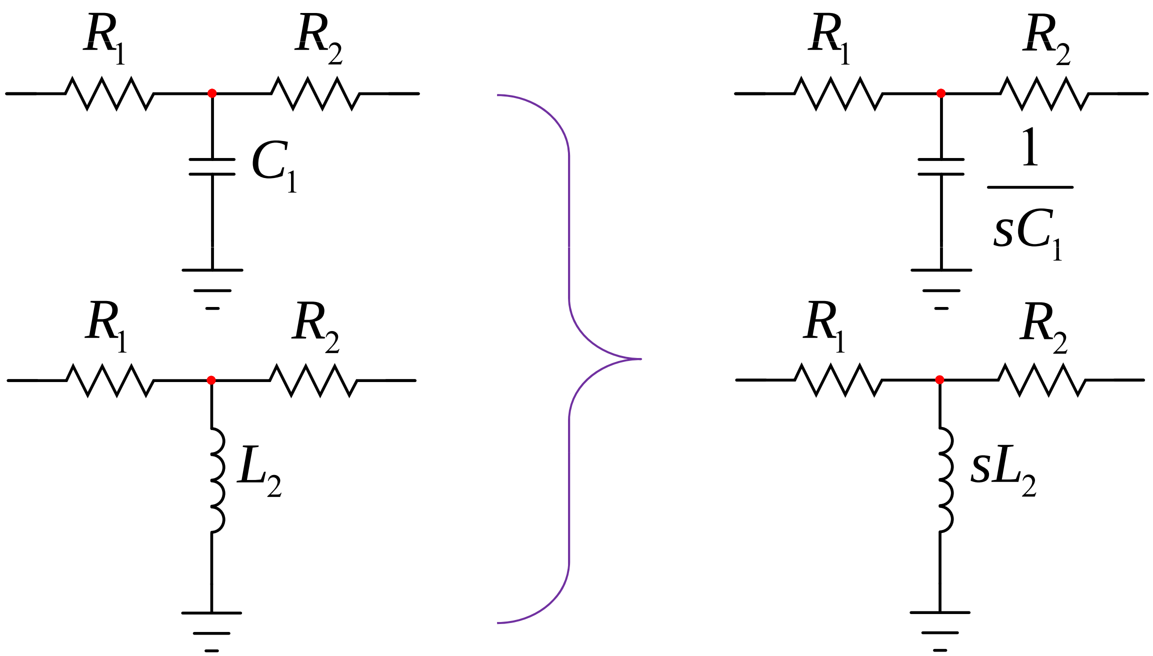


A parallel impedance
shorts the path to ground

- What is a *transformed* network?

The Transformed Network

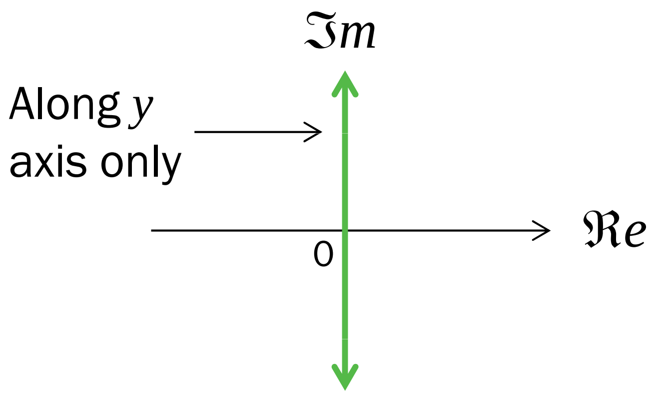
- Reactances are replaced by their Laplace expression



- The circuit is then observed at the zero frequency

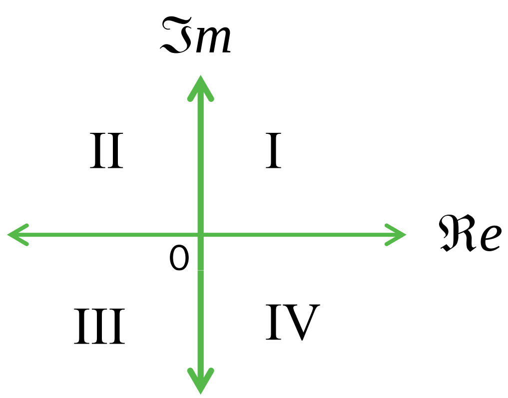
Harmonic Analysis

- Harmonic analysis is performed for $s = j\omega$



- ❖ imaginary frequencies only
- ❖ no real negative frequencies

- In the transformed network, consider $s = \sigma + j\omega$



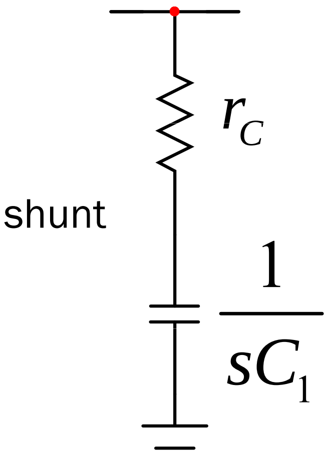
The four quadrants are considered!

- ❖ negative angular frequencies
- ❖ real or imaginary ang. frequencies

- There is no physical meaning: mathematical abstraction!

Considering a Negative Frequency

- For $s = s_{z1}$, the RC impedance is a short circuit

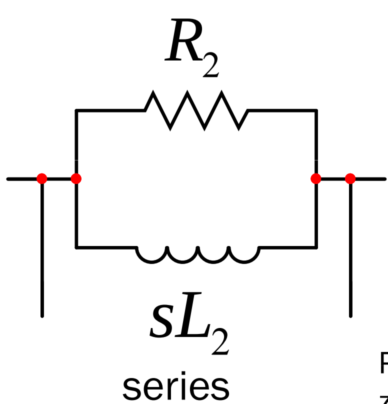


$$Z_1(s) = \frac{\underbrace{\boxed{1 + sr_C C_1}}^{=0}}{sC_1}$$

$$Z_1\left(s_{z_1}\right) = 0\ \Omega$$

$$s_{z_1} = -\frac{1}{r_C C_1}$$

- For $s = s_{z2}$, the RL impedance is infinite



$$Z_2(s) = \frac{sL_2 R_2}{\underbrace{\boxed{R_2 + sL_2}}_{=0}}$$

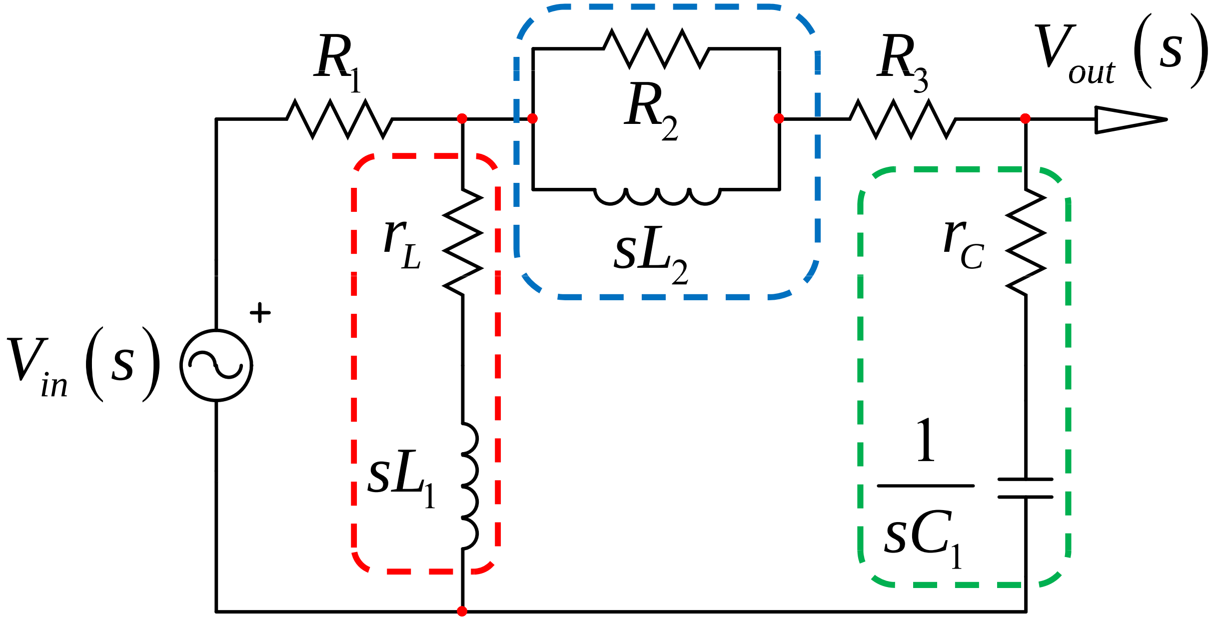
Poles of the *RL* network become zeros of the transfer function.

$$Z_2\left(s_{z_2}\right) \to \infty\ \Omega$$

$$s_{z_2} = -\frac{R_2}{L_2}$$

Zeros by Inspection

- Identify *transformed* open circuits/short circuits



$$s_{z_1} = -\frac{r_L}{L_1} \qquad \omega_{z_1} = \frac{r_L}{L_1}$$

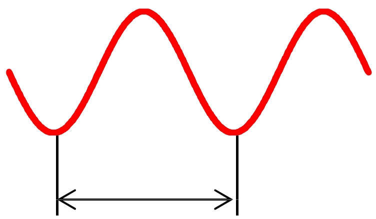
$$s_{z_2} = -\frac{R_2}{L_2} \qquad \omega_{z_2} = \frac{R_2}{L_2}$$

$$s_{z_3} = -\frac{1}{r_C C_1} \qquad \omega_{z_3} = \frac{1}{r_C C_1}$$

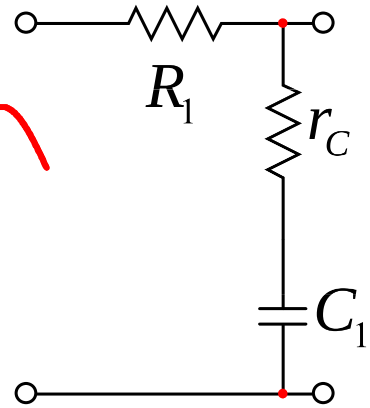
$$N(s) = \left(1+\frac{s}{\omega_{z_1}}\right)\left(1+\frac{s}{\omega_{z_2}}\right)\left(1+\frac{s}{\omega_{z_3}}\right)$$

A Zero in the Laboratory

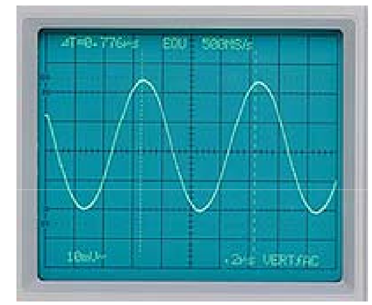
- ❑ Can you observe a *zero* in the lab?



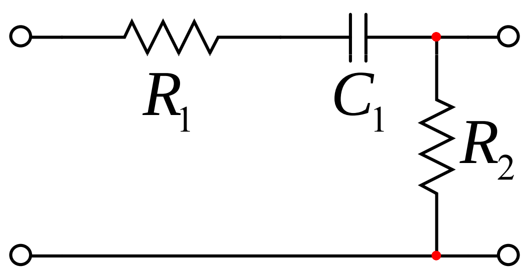
$T = 2\pi r_C C_1$



response is non-zero



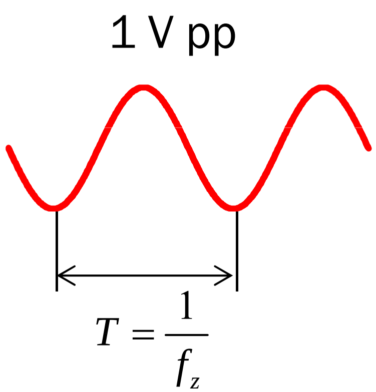
- ❑ No, because this is a harmonic analysis $s = j\omega$
- ✓ It works for a zero at the origin: dc block

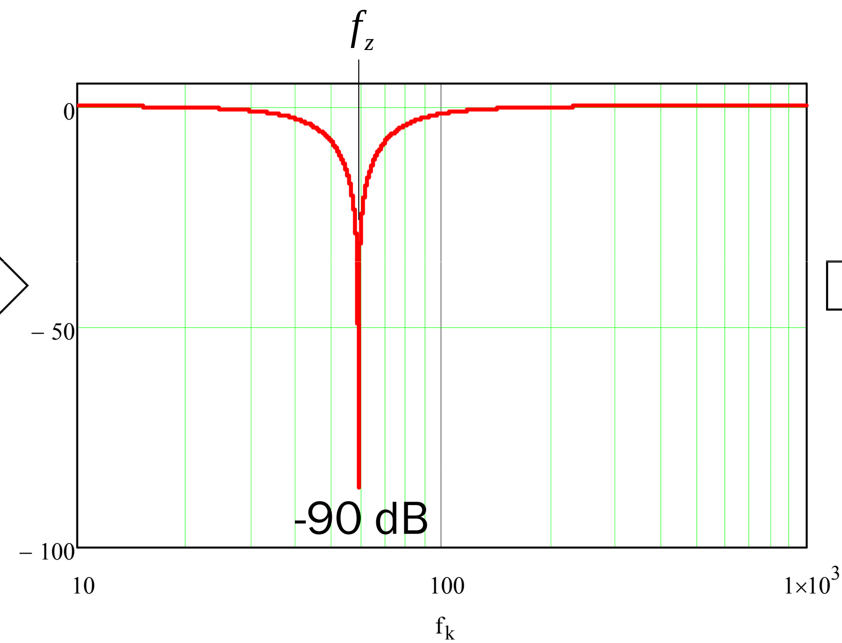


$$V_{out}\left(0\text{ Hz}\right) = 0\text{ V}$$

A Notch Truly Nulls the Response

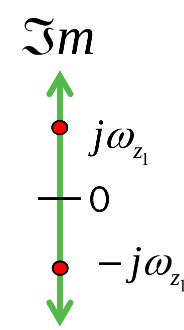
- When Q approaches infinity, zeros become imaginary





Observable null

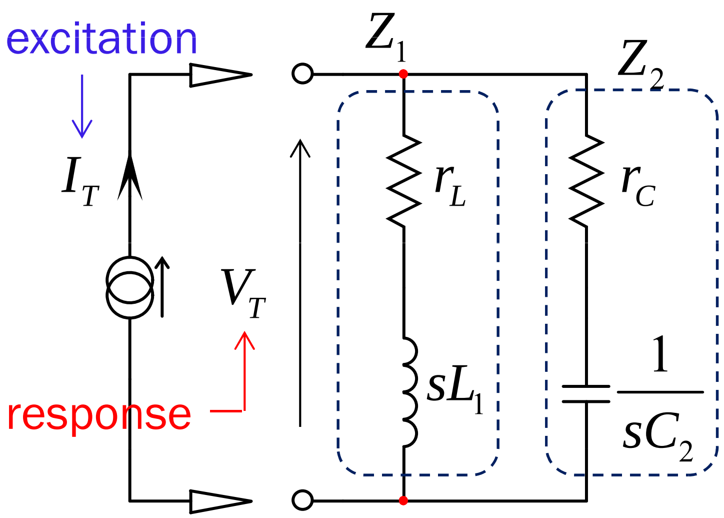
$$V_{out}(f_z) = 31\,\mu V$$



- Build a high-Q notch and you can observe the null
  - Roots are along the y axis: harmonic analysis

Find the Zeros by Inspection

- When does the response disappear?



$$Z_1(s) = r_L + sL_1 = 0 \quad s_{z_1} = -\frac{r_L}{L_1} \quad \omega_{z_1} = \frac{r_L}{L_1}$$

$$Z_2(s) = r_C + \frac{1}{sC_2} = 0 \quad s_{z_2} = -\frac{1}{r_C C_2} \quad \omega_{z_2} = \frac{1}{r_C C_2}$$

- The numerator is obtained without algebra

$$N(s) = \left(1 + \frac{s}{\omega_{z_1}}\right)\left(1 + \frac{s}{\omega_{z_2}}\right)$$

Inspection gives the simplest expressions

Course Agenda

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- ❑ **The Null Double Injection**
- ❑ 2nd-Order Networks
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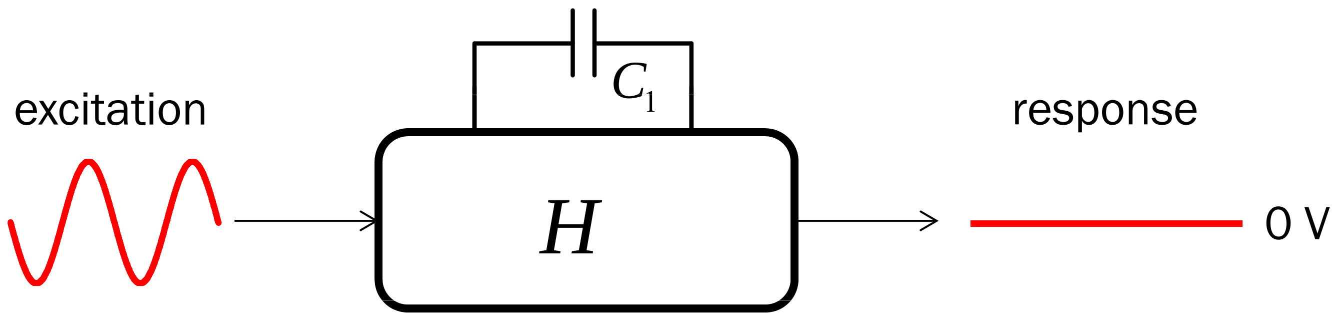
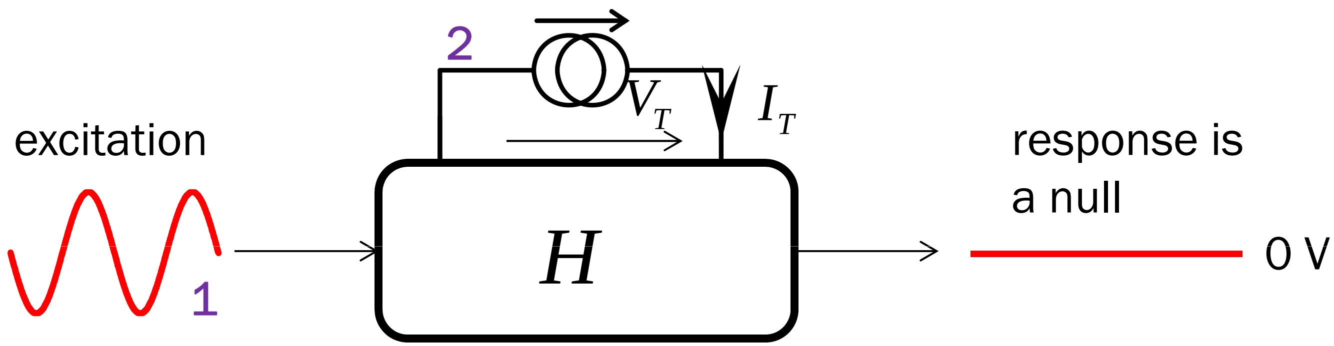
- ❑ A CCM Buck-Boost in Voltage Mode

The Null Double Injection

- A null implies an injection but no response



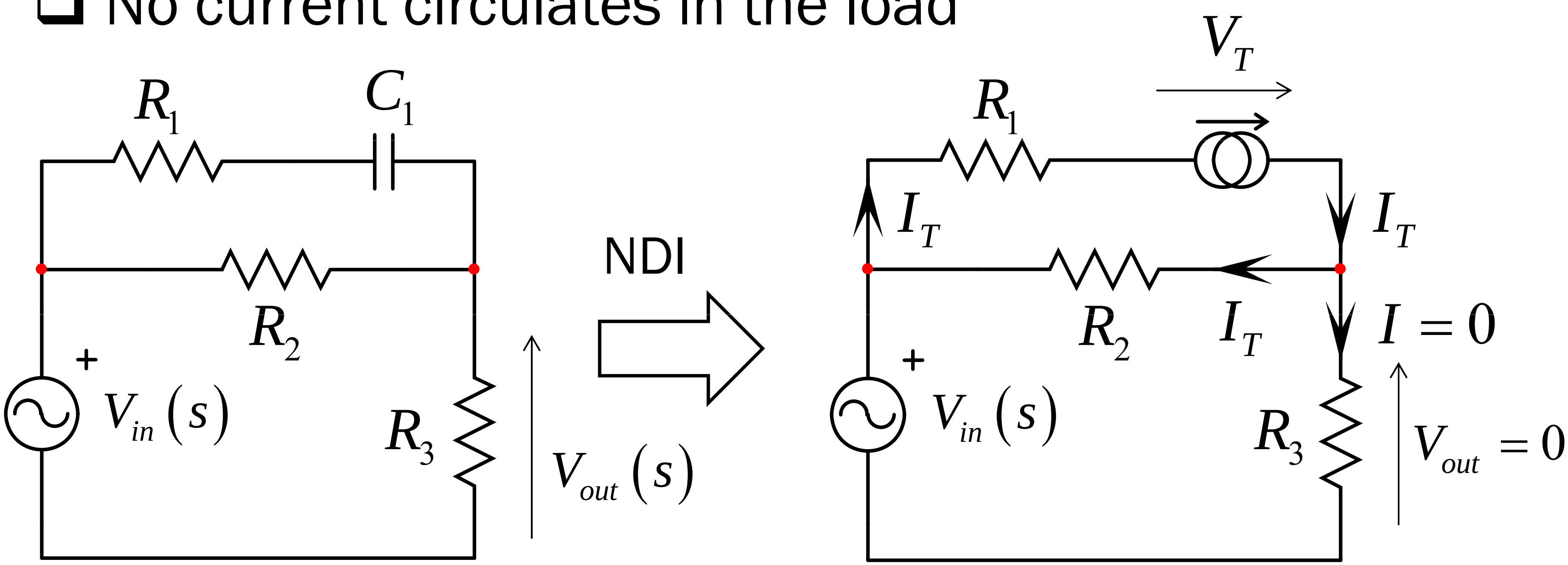
- What is the time constant in this mode? $\tau = \frac{V_T}{I_T} C_1$



- Double injection with a nulled response (NDI)

What is a Null in the Response?

- No current circulates in the load

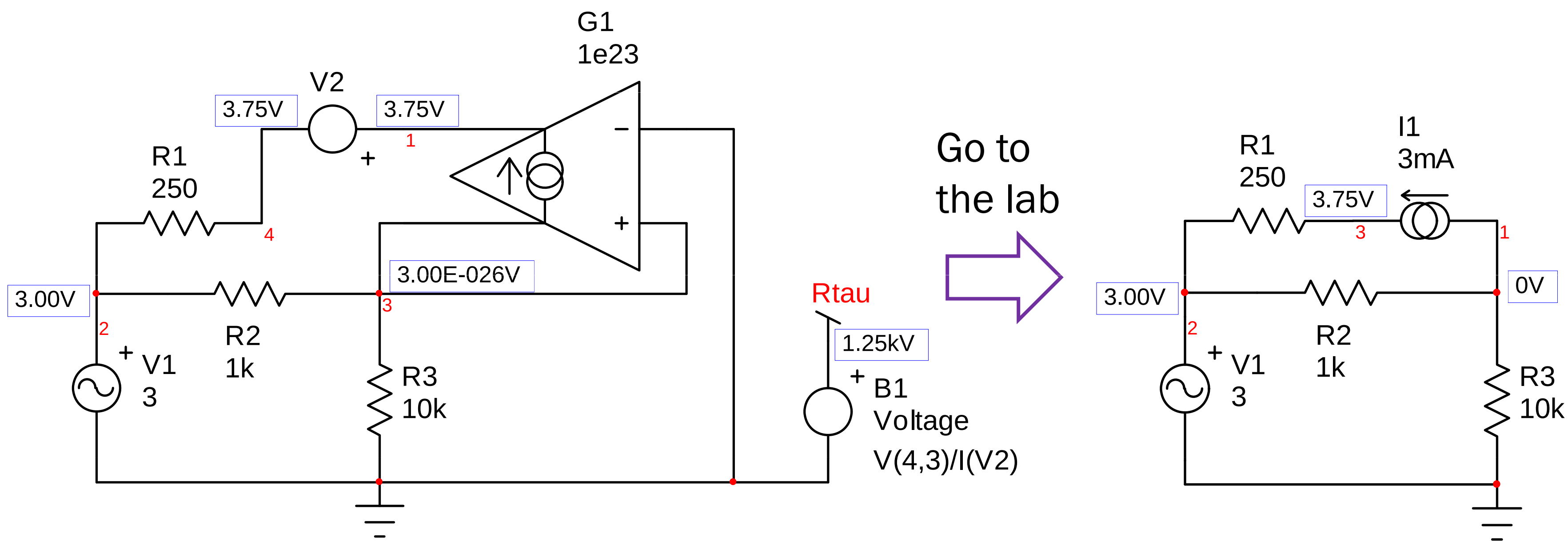


- In this configuration, the resistance is $R_1 + R_2$

$$\tau = (R_1 + R_2)C_1 \quad \text{and} \quad N(s) = 1 + s(R_1 + R_2)C_1$$

Does it Have a Physical Meaning?

- A certain combination of V and I cancels the response

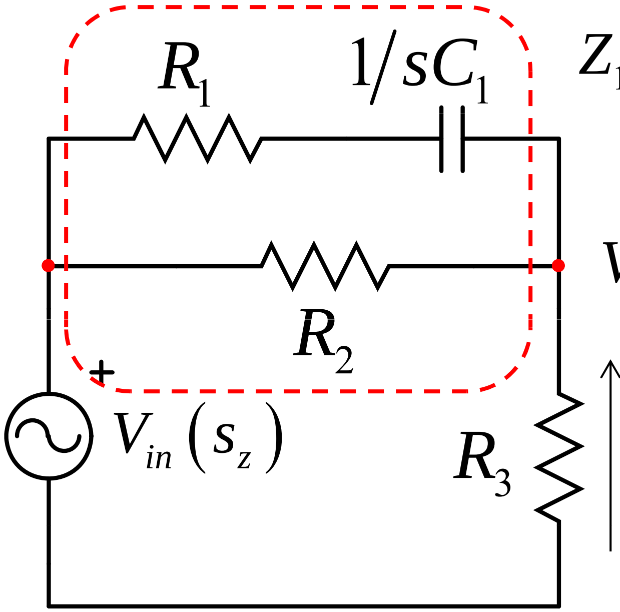


- The current source G_1 adjusts to set V_{out} to 0 V: NDI

$$R = \frac{V_T}{I_T} = \frac{3.75\ \text{V}}{3\ \text{mA}} = 1250\ \Omega \longrightarrow R_1 + R_2$$

Inspection Would Work Here as Well

- What prevents the excitation from building a response?



$$Z_1(s_z) \to \infty$$

$$V_{out}(s_z) = 0$$

$$Z_1(s) = \frac{N(s)}{D(s)} \to \infty$$

$$\longrightarrow D(s) = 0$$

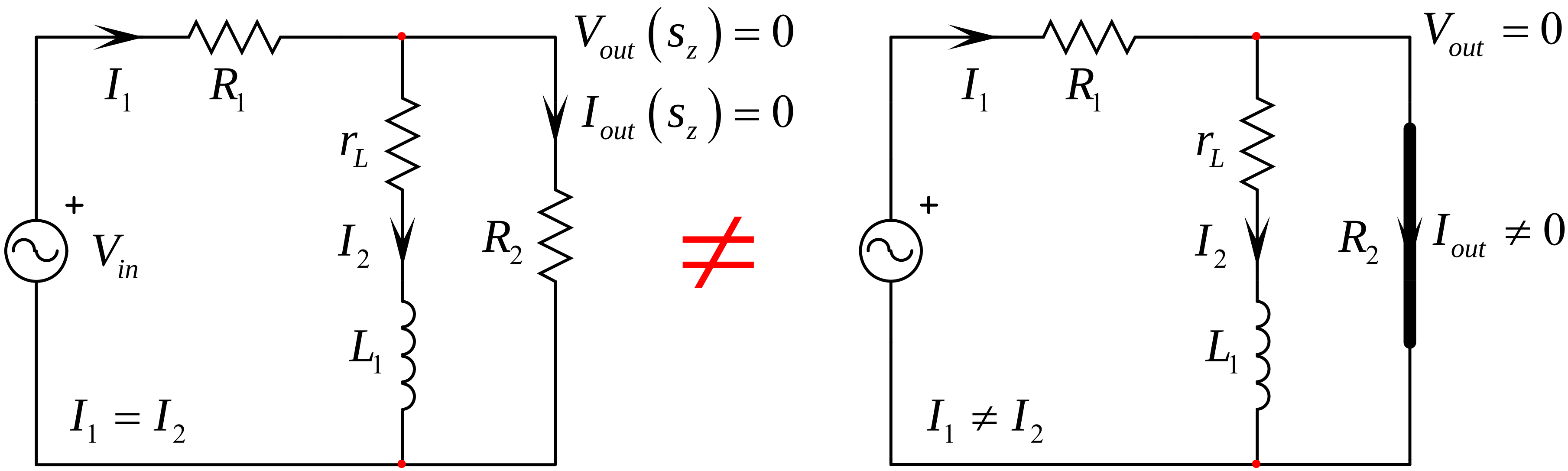
- What is the denominator of Z_1? Look at R driving C_1

$$R = R_1 + R_2 \longrightarrow D(s) = 1 + sC_1(R_1 + R_2) \longrightarrow \omega_p = \frac{1}{C_1(R_1 + R_2)}$$

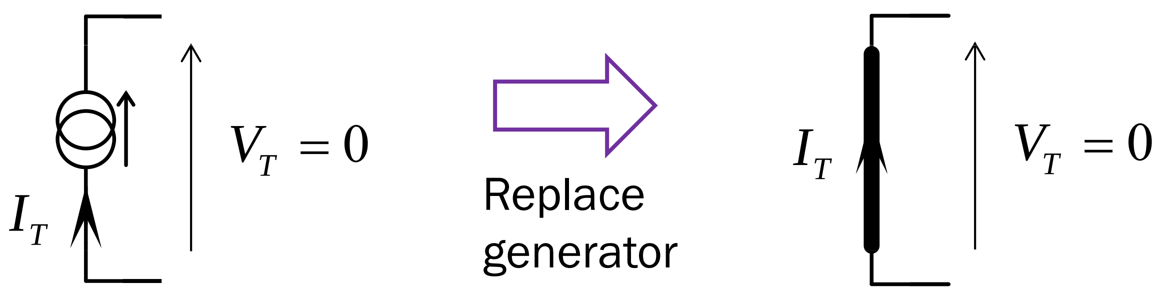
- Z_1's pole is H's zero

A Null is Not a Short Circuit

- See the output null as a virtual ground: no short!



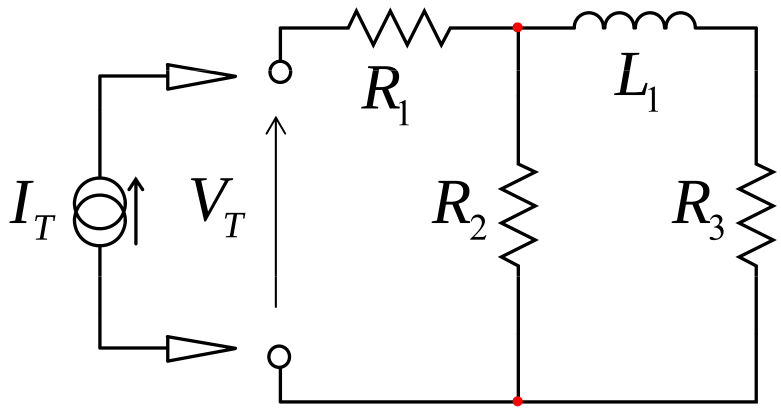
- 0 V across a current generator is a true short circuit



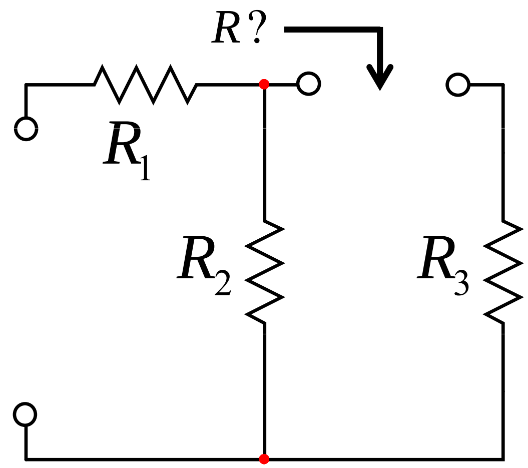
Degenerate case

Degenerate Case Applied to Impedance

- Determine the input impedance of this circuit

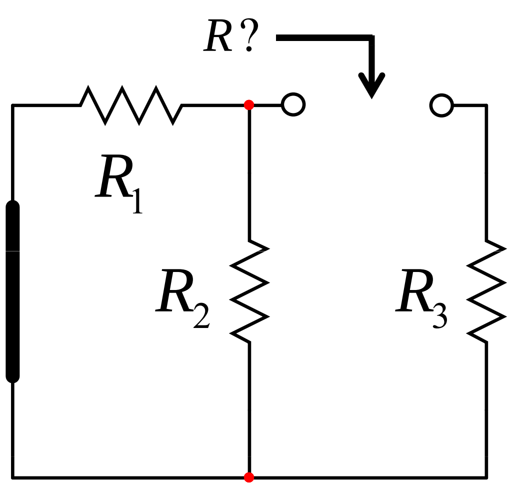


Set I_T to 0 A



$$\tau = \frac{L_1}{R_2 + R_3}$$

NDI, $V_T = 0$
short source



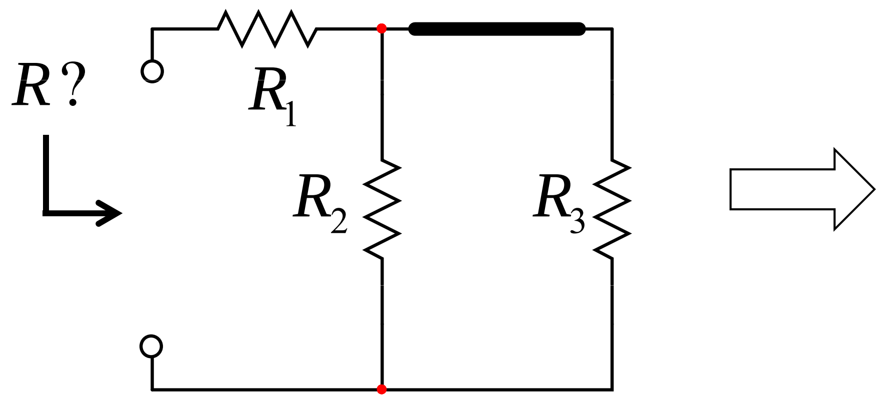
$$\tau = \frac{L_1}{R_2 \parallel R_1 + R_3}$$

$$D(s) = 1 + s\frac{L_1}{R_2 + R_3}$$

$$N(s) = 1 + s\frac{L_1}{R_2 \parallel R_1 + R_3}$$

Three Steps for the Transfer Function

- For s = 0, replace the inductor by a short circuit



$$R_0 = R_1 + R_2 \,||\, R_3$$

- Result is well ordered and obtained without KVL/KCL

$$Z(s) = R_0 \frac{1 + \frac{s}{\omega_z}}{1 + \frac{s}{\omega_p}}$$

$$R_0 = R_1 + R_2 \,||\, R_3$$

$$\omega_p = \frac{R_2 + R_3}{L_1} \qquad \omega_z = \frac{R_2 \,||\, R_1 + R_3}{L_1}$$

Summary for 1st-order Systems - I

- ❑ Observe the circuit for $s = 0$
  - ❖ short inductor, open capacitor
  - ✓ You have H_0

- ❑ Turn the excitation off
  - ❖ voltage source is replaced by a short circuit
  - ❖ current source is open-circuited

- ❑ Remove the energy storage element
- ❑ Determine the resistance R_D looking into its terminals

$$\Rightarrow \quad \tau_D = R_D C \quad \text{or} \quad \tau_D = \frac{L}{R_D}$$

Summary for 1st-order Systems - II

- Bring excitation source back in place
- Null the output, $V_{out} = 0$ V and $I_{out} = 0$ A
- Determine R_N driving the energy-storage component

$$\tau_N = R_N C \quad \text{or} \quad \tau_N = \frac{L}{R_N}$$

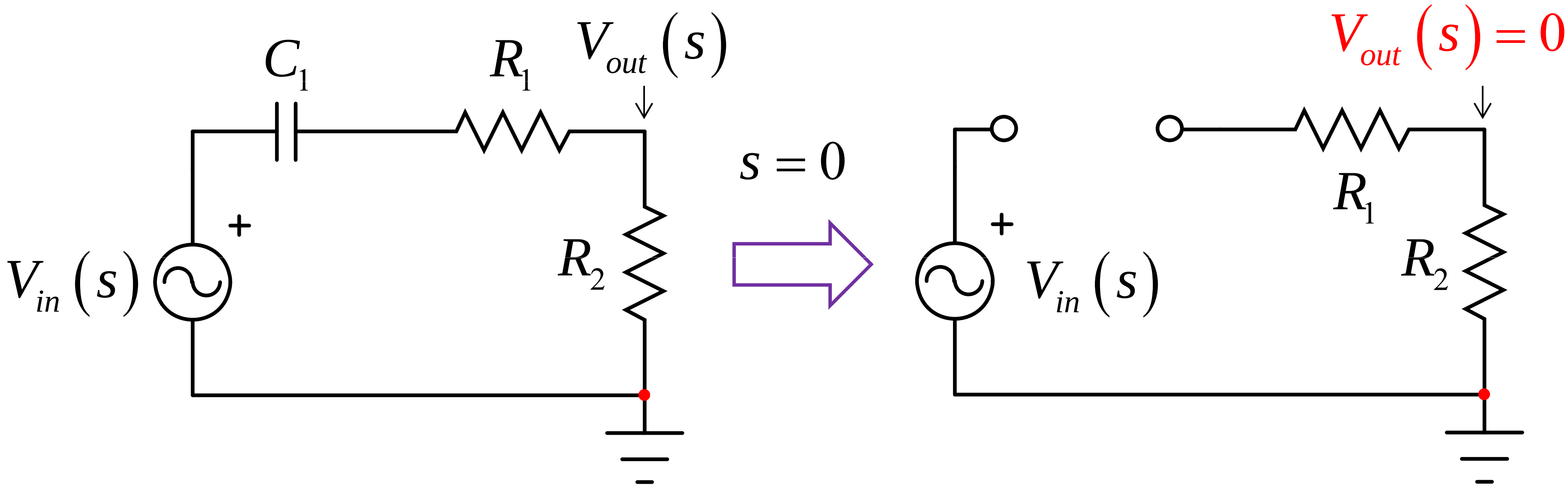
- Combine time constants and dc gain

$$H(s) = H_0 \frac{1 + s\tau_N}{1 + s\tau_D} = H_0 \frac{1 + \frac{s}{\omega_z}}{1 + \frac{s}{\omega_p}}$$

If possible, use inspection: simplest possible form

What if dc Gain Does not Exist?

- ❑ If you have a series capacitor

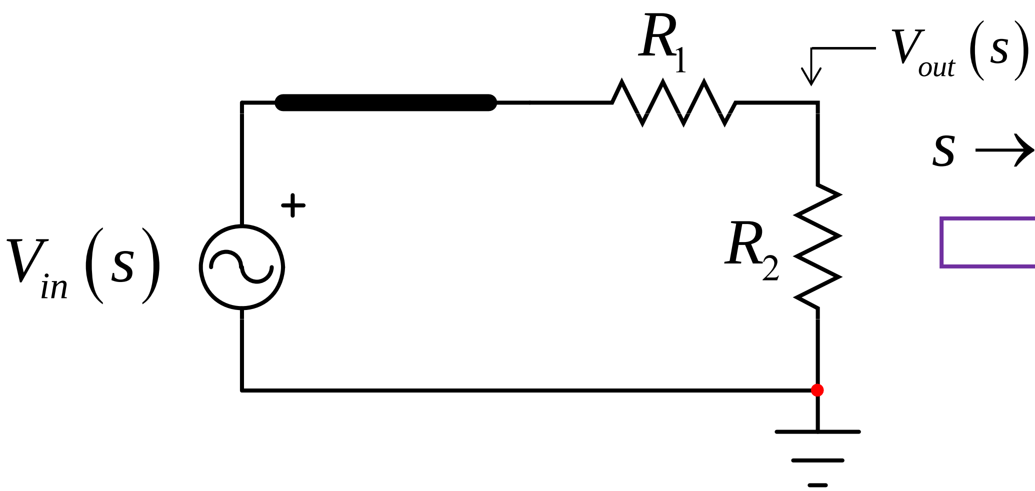


- ❑ The dc or quasi-static gain is 0

$$H_0 = 0 \xrightarrow{\text{No longer applies}} H(s) = H_0 \frac{1 + s\tau_N}{1 + s\tau_D}$$

Consider High-Frequency Model

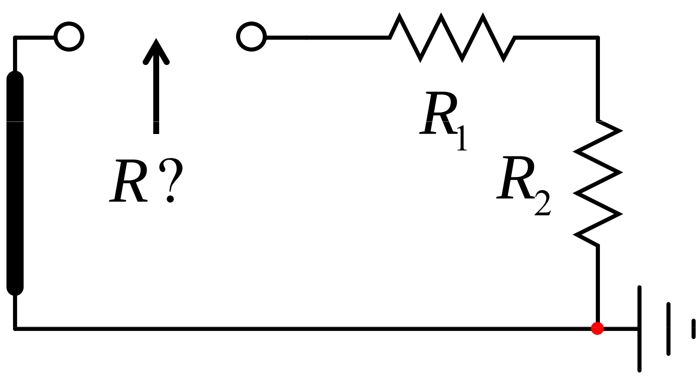
- Rather than considering s at 0, consider $s \longrightarrow \infty$



$s \rightarrow \infty$

$$H_\infty = \frac{R_2}{R_1 + R_2}$$

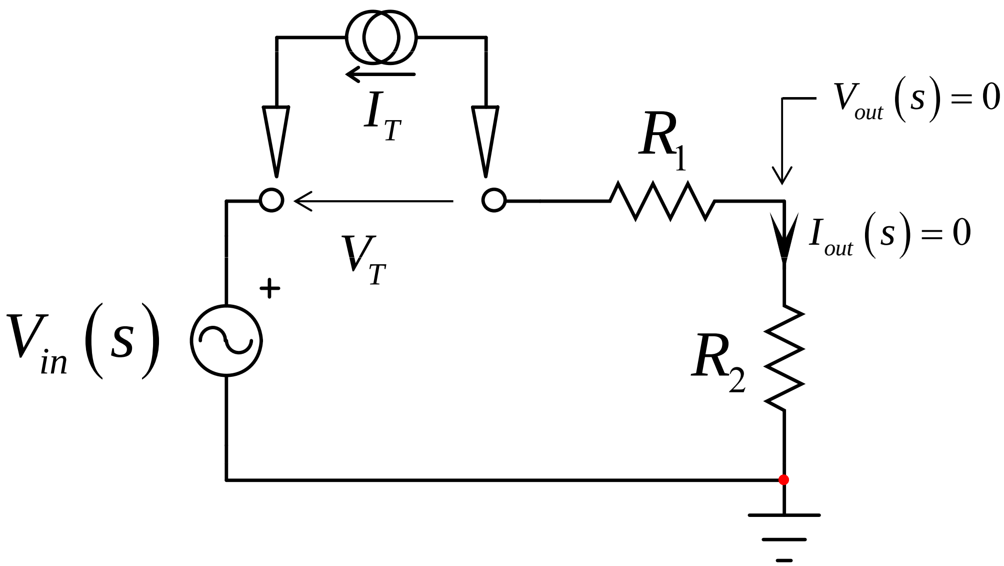
- Look at the resistance driving C while excitation is off



$$\tau = (R_1 + R_2)C_1$$

Null the Response to Get the Zero

- Is there a zero other than at the origin?



$$I_{out}(s)=0 \longrightarrow I_{R_1}=0$$

$$I_T=0 \longrightarrow \frac{V_T}{I_T}\to\infty$$

$$\Rightarrow \quad \tau_N\to\infty$$

- We have a high-frequency gain and two time constants

$$H_\infty=\frac{R_2}{R_1+R_2} \qquad \tau_D=(R_1+R_2)C_1 \qquad \tau_N\to\infty$$

Two Formulas for the Same Function

- The Extra Element Theorem shows that

$$H(s) = H_0 \frac{1+s\tau_N}{1+s\tau_D} \quad \text{is equivalent to} \quad H(s) = H_\infty \frac{1+\dfrac{1}{s\tau_N}}{1+\dfrac{1}{s\tau_D}}$$

- Time constants are similar in both expressions

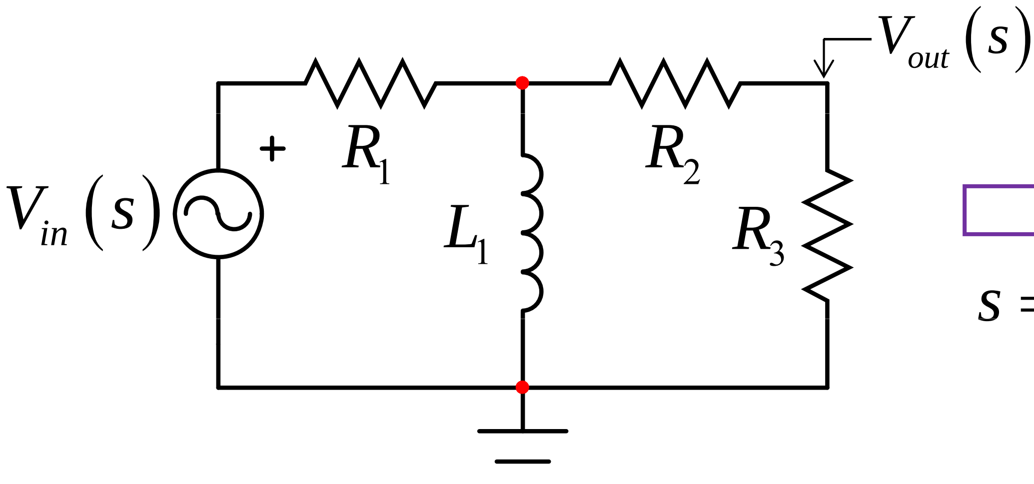
$$H(s) = \frac{R_2}{R_1+R_2} \frac{1+\dfrac{1}{s\cdot\infty}}{1+\dfrac{1}{sC_1(R_1+R_2)}} = \frac{R_2}{R_1+R_2} \frac{1}{1+\dfrac{1}{sC_1(R_1+R_2)}}$$

- It is a low-entropy form featuring an inverted pole

R. D. Middlebrook, "Null Double Injection and the Extra Element Theorem", IEEE Transactions on on Education, Vol. 32, NO. 3, August 1989.

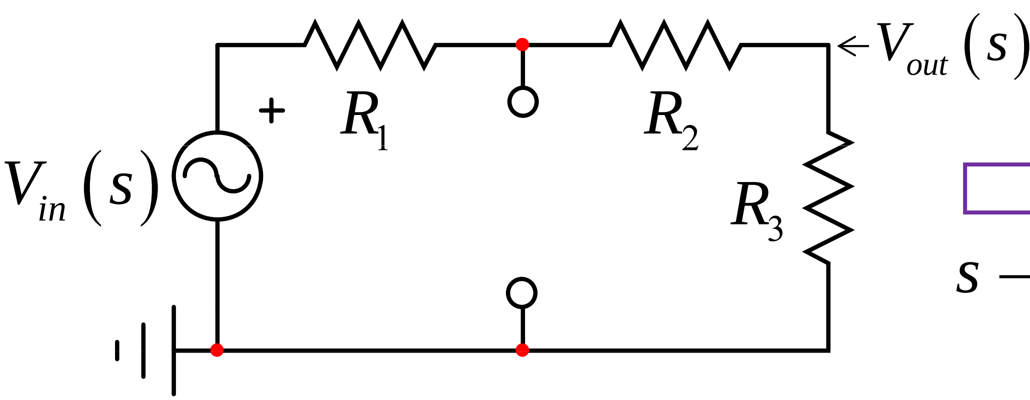
Another Example

- The inductance is a short circuit in dc



$$s = 0 \Rightarrow H_0 = 0$$

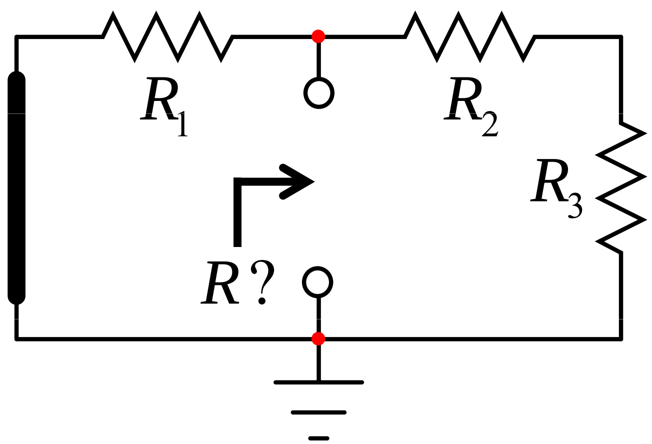
- Consider the circuit at high frequency instead



$$s \to \infty \Rightarrow H_\infty = \frac{R_3}{R_3 + R_2 + R_1}$$

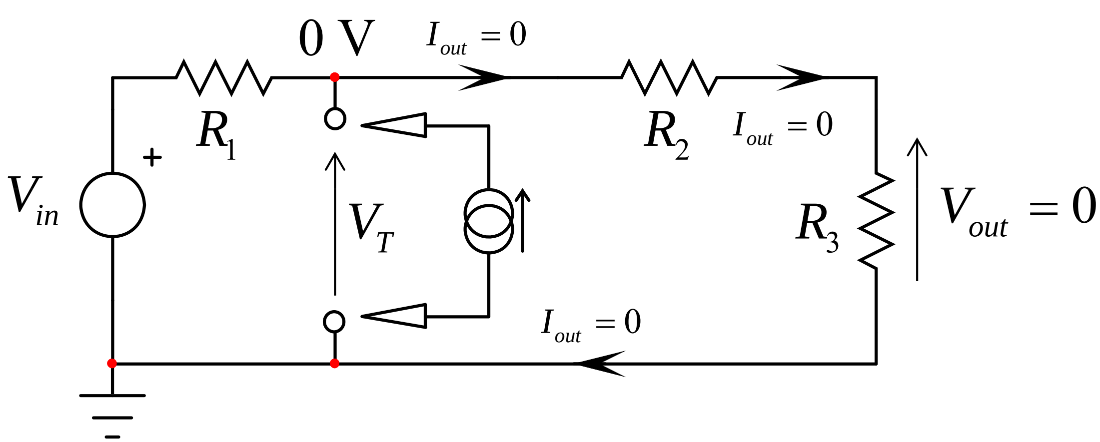
Time Constant Involving Inductor

- Look at the inductor time constant while V_{in} is 0 V



$$\tau_D = \frac{L_1}{R_1 \| (R_2 + R_3)}$$

- Now consider a null output voltage

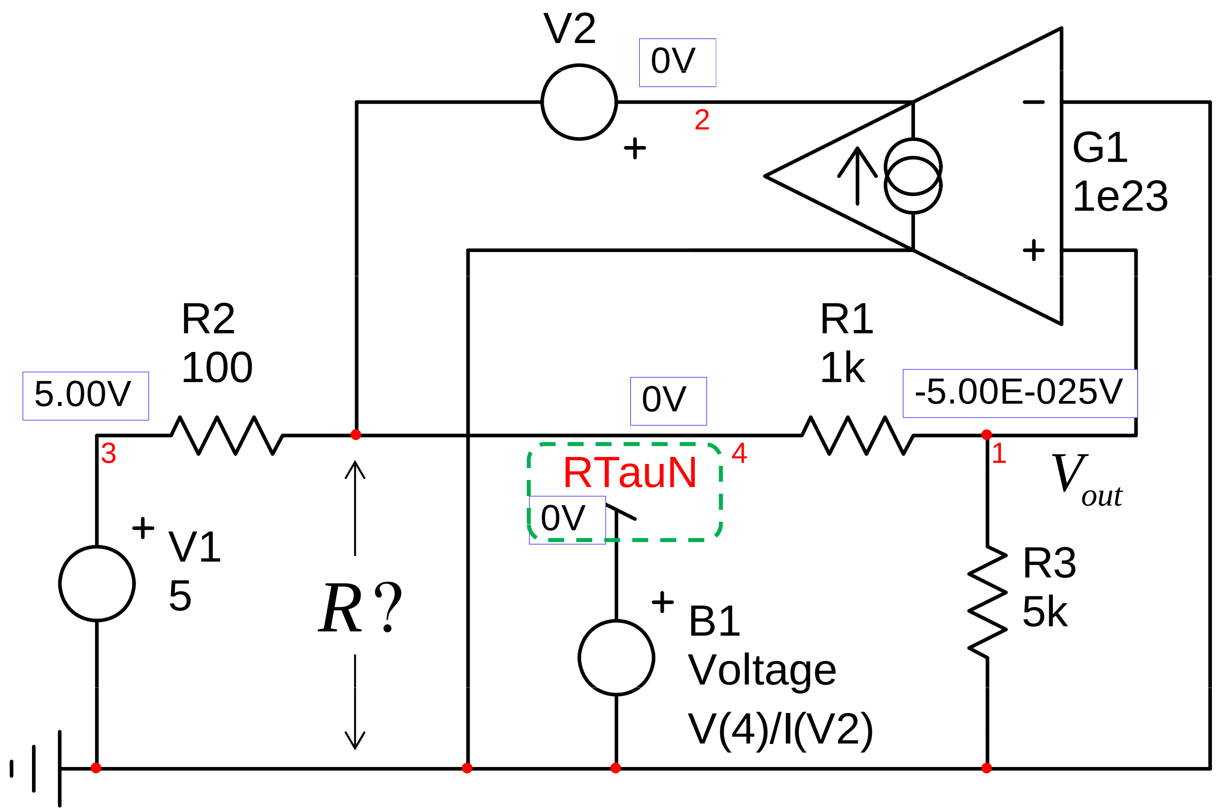


$$\frac{V_T}{I_T} = \frac{0}{I_T} = 0$$

$$\tau_N = \frac{L_1}{R} \to \infty$$

Determining the Zero

- Check with SPICE if a doubt exists



$$R = 0$$

$$\tau_N = \frac{L_1}{R} \rightarrow \infty$$

- G_1 injects a current to maintain V_{out} at 0: NDI

Final Transfer Function

- Assemble time constants to form $H(s)$

$$H(s)=\frac{R_3}{R_3+R_2+R_1}\frac{1+\frac{1}{s\cdot\infty}}{1+\frac{1}{s\frac{L_1}{R_1\|\left(R_2+R_3\right)}}}=\frac{R_3}{R_3+R_2+R_1}\frac{1}{1+\frac{1}{s\frac{L_1}{R_1\|\left(R_2+R_3\right)}}}$$

- Rewrite the expression in a compact form

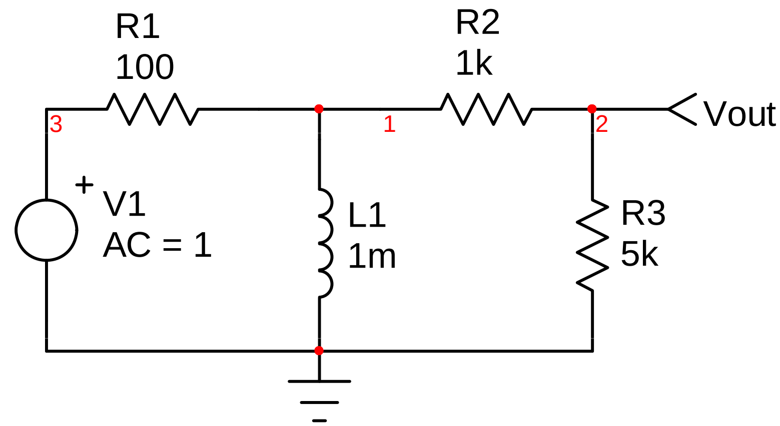
$$H(s)=H_\infty\frac{1}{1+\frac{\omega_p}{s}}\qquad H_\infty=\frac{R_3}{R_3+R_2+R_1}\qquad \omega_p=\frac{R_1\|\left(R_2+R_3\right)}{L_1}$$

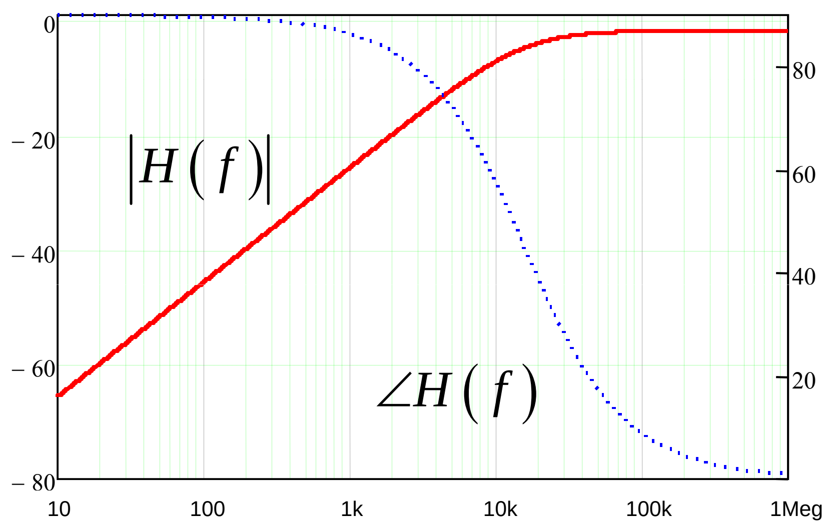
Check with Mathcad and SPICE

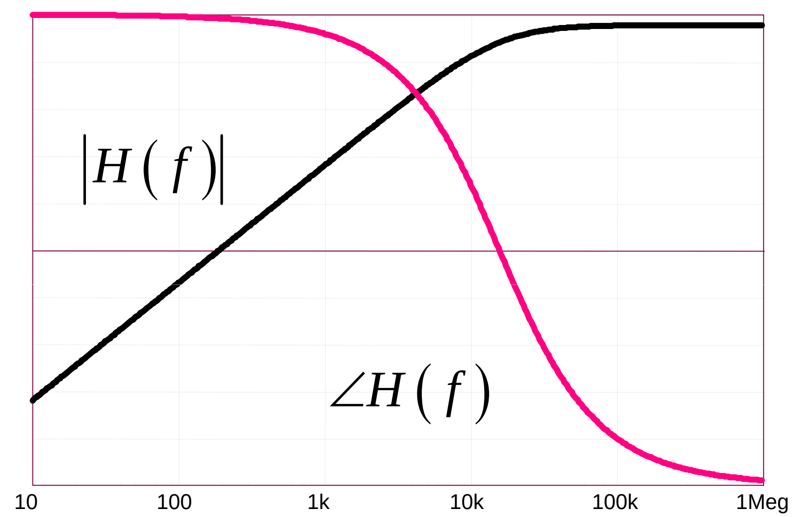
$R_1 := 100\Omega \qquad R_2 := 1k\Omega \qquad R_3 := 5k\Omega \qquad L_1 := 1mH \qquad \|(x,y) := \frac{x \cdot y}{x + y}$

$$\tau_D := \frac{L_1}{R_1 \| (R_2 + R_3)} = 10.167\,\mu s \qquad H_{inf} := \frac{R_3}{R_1 + R_2 + R_3} = 0.82$$

$$H_1(s) := H_{inf} \cdot \frac{1}{1 + \frac{1}{s \cdot \tau_D}} \qquad f_p := \frac{1}{2\pi \cdot \tau_D} = 15.655kHz$$

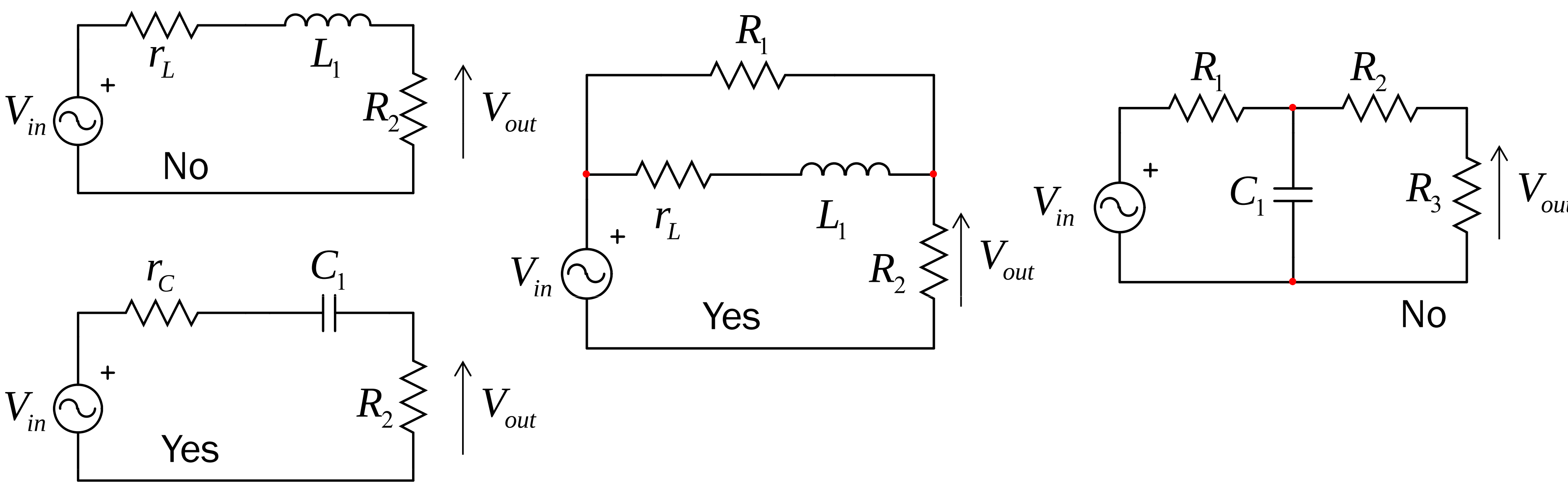






Checking for a Zero

- ❑ Is there a quick way to check if there is a zero?
  - ➢ Yes! Put the reactance in its high-frequency state
  - ➢ Check if the response is still there
  - ✓ If yes, there is a zero associated with the reactance
  - ✓ If not, there is no zero in the circuit



Course Agenda

- ❑ What is a Transfer Function?
- ❑ Why do We Need New Analytical Techniques?
- ❑ Time Constants and Poles
- ❑ Identifying the Zeros
- ❑ The Null Double Injection
- ❑ 2nd-Order Networks
- ❑ The PWM Switch Model
- ❑ A CCM Buck in Voltage Mode
- ❑ A CCM Buck-Boost in Voltage Mode

Fractions and Dimensions

- A 1st-order system follows the form

$$H(s)=\frac{N(s)}{D(s)}=\frac{a_0+a_1 s}{b_0+b_1 s}\xrightarrow{\text{factoring}}H(s)=\frac{a_0}{b_0}\frac{1+\frac{a_1}{a_0}s}{1+\frac{b_1}{b_0}s}$$

- Leading term (if any) carries the unit

$$\underset{[\Omega]}{Z(s)}=\underset{[\Omega]}{R_0}\underbrace{\boxed{\frac{1+\frac{a_1}{a_0}s}{1+\frac{b_1}{b_0}s}}}_{\text{Unitless}}\Rightarrow\left.\underbrace{\begin{matrix}1+\frac{a_1}{a_0}s\\ \Updownarrow\\ 1+\frac{b_1}{b_0}s\end{matrix}}_{\text{Unitless}}\right\}\begin{matrix}\frac{a_1}{a_0}\rightarrow[\mathrm{s}]\rightarrow\tau_N\\ \\ \frac{b_1}{b_0}\rightarrow[\mathrm{s}]\rightarrow\tau_D\end{matrix}$$

2nd-Order System

- A 2nd-order system follows the form

$$H(s)=\frac{\alpha_0+\alpha_1 s+\alpha_2 s^2}{\beta_0+\beta_1 s+\beta_2 s^2}\xrightarrow[\text{Factoring } \beta_0]{\text{Factoring } \alpha_0} H(s)=H_0\boxed{\frac{1+a_1 s+a_2 s^2}{1+b_1 s+b_2 s^2}}$$

Unitless

Carries the unit

- The second fraction is unitless

$$a_1=\frac{\alpha_1}{\alpha_0}\to[\mathrm{s}]\to\tau_{1N}+\tau_{2N} \qquad a_2=\frac{\alpha_2}{\alpha_0}\to\left[\mathrm{s}^2\right]\to\tau_{1N}\tau_{2N}^{1}\ \text{or}\ \tau_{2N}\tau_{1N}^{2}$$

sum

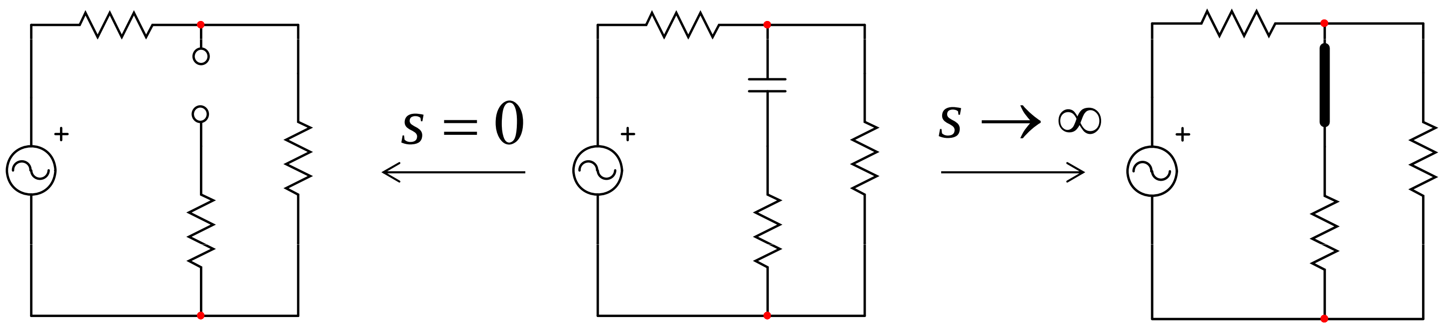
product

$$b_1=\frac{\beta_1}{\beta_0}\to[\mathrm{s}]\to\tau_{1D}+\tau_{2D} \qquad b_2=\frac{\beta_2}{\beta_0}\to\left[\mathrm{s}^2\right]\to\tau_{1D}\tau_{2D}^{1}\ \text{or}\ \tau_{2D}\tau_{1D}^{2}$$

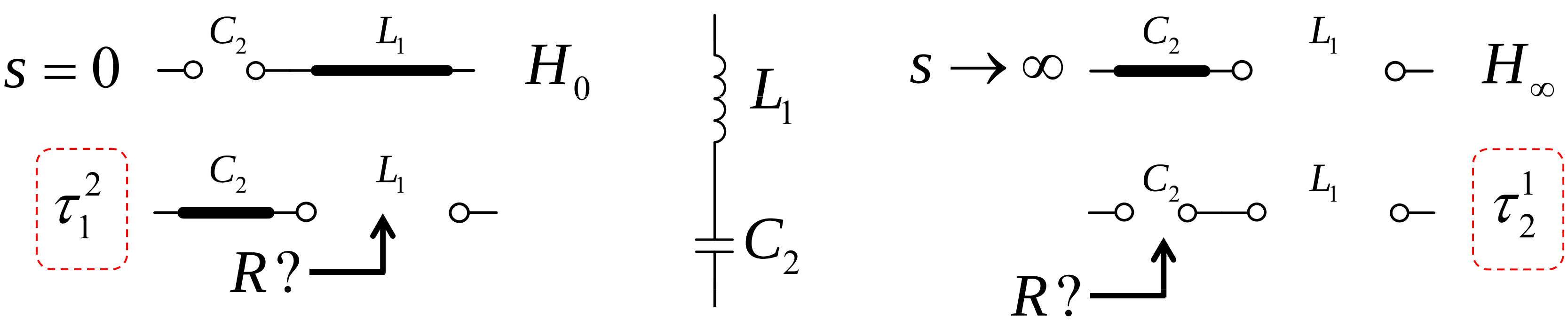
reactance 1 reactance 2

Alternating the Reactance States

- ❑ In a 1st-order circuit, there is one reactance
  - ❖ it is either in a high-frequency state or in a dc state



- ❑ In a 2nd-order circuit, there are two reactances
  - ❖ we can consider individual states



Introducing the Notation

- Set one reactance into its high-frequency state

$$\tau_{2}^{1}$$

1 → Reactance 1 is in its high-frequency state

2 → What resistance drives reactance 2?

$R?$

$$\tau_{1}^{2}$$

2 → Reactance 2 is in its high-frequency state

1 → What resistance drives reactance 1?

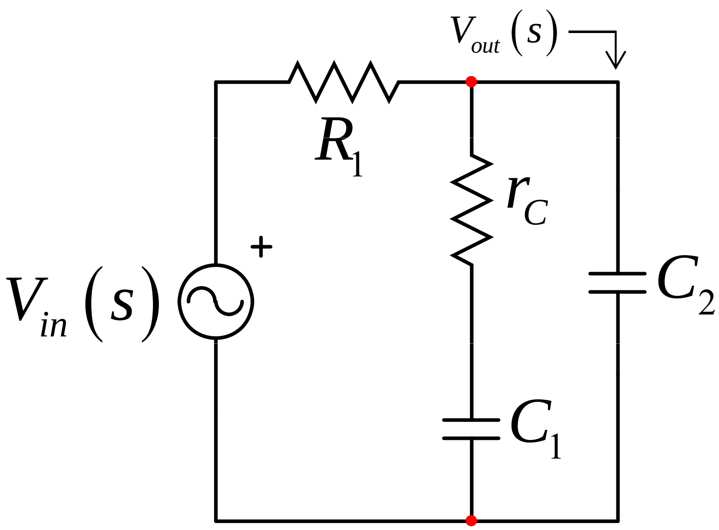
$R?$

- There is redundancy: pick the simplest result

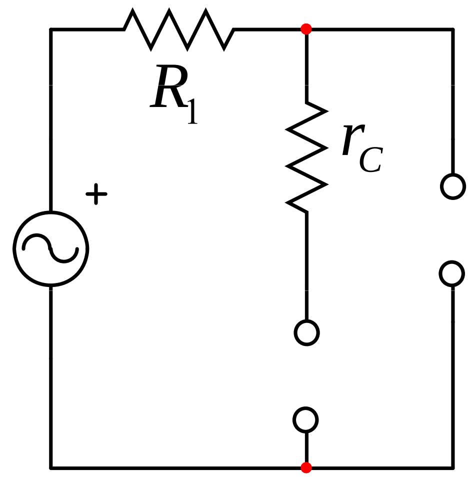
$$b_2 = \tau_1 \tau_2^1 \Longleftrightarrow b_2 = \tau_2 \tau_1^2$$

Example with Capacitors

- Assume the following 2-capacitor circuit

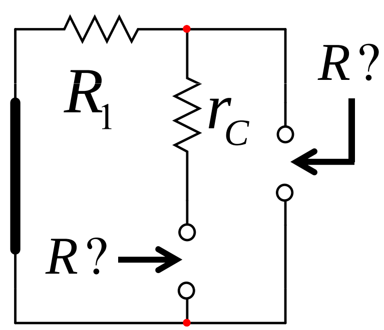


$s = 0$



$H_0 = 1$

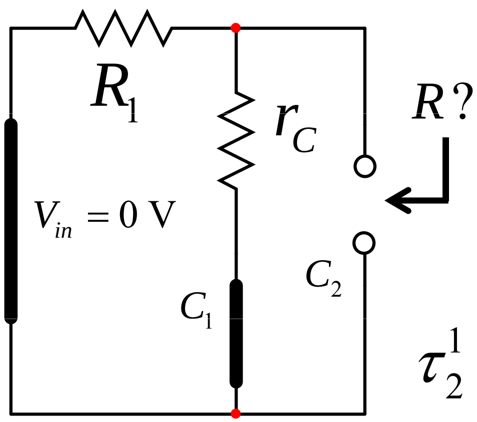
- Determine the two time constants while V_{in} is 0 V



$$\left.\begin{aligned} \tau_1 &= C_1\left(r_C + R_1\right) \\ \tau_2 &= C_2 R_1 \end{aligned}\right\} \quad b_1 = C_1\left(r_C + R_1\right) + C_2 R_1$$

Determining the Higher-Order Term

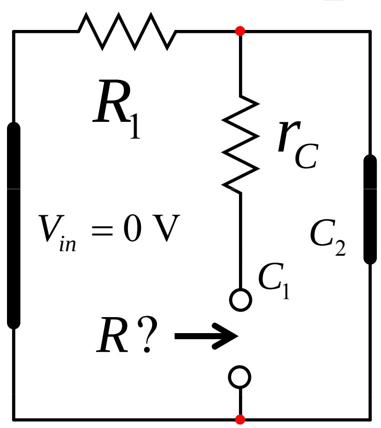
- Place C_1 in its high-frequency and look into C_2



$$b_2 = \tau_1\tau_2^1 = C_1\left(r_C + R_1\right)C_2\left(R_1 \parallel r_C\right)$$

$$\tau_2^1 = \left(R_1 \parallel r_C\right)C_2$$

- Place C_2 in its high-frequency and look into C_1



$$b_2 = \tau_2\tau_1^2 = C_2R_1C_1r_C$$

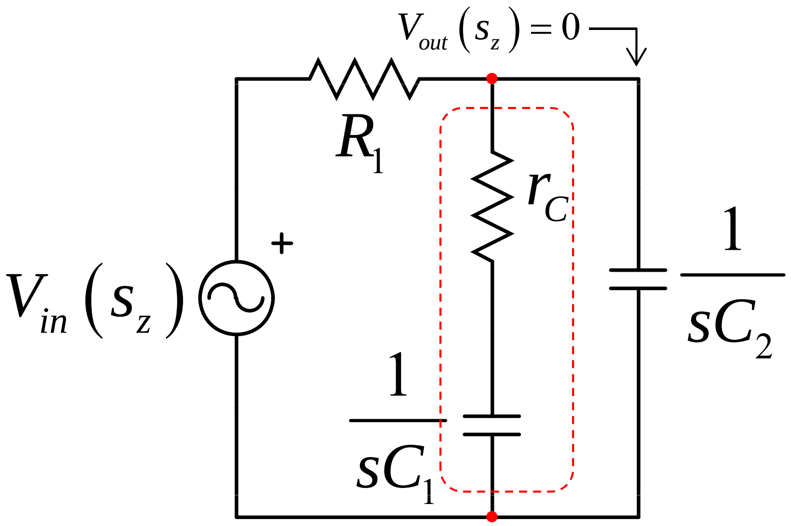
$$\tau_1^2 = r_CC_1$$

Denominator is Completed

- The denominator can be assembled

$$D(s) = 1 + b_1 s + b_2 s^2 = 1 + \left[C_1\left(r_C + R_1\right) + C_2 R_1\right]s + C_2 R_1 C_1 r_C s^2$$

- Is there a zero in this network?



$$r_C + \frac{1}{sC_1} = 0 \quad \Rightarrow \quad s_{z_1} = -\frac{1}{r_C C_1} \qquad \omega_{z_1} = \frac{1}{r_C C_1}$$

$$H(s) = \frac{1 + s r_C C_1}{1 + \left[C_1\left(r_C + R_1\right) + C_2 R_1\right]s + C_2 R_1 C_1 r_C s^2}$$

No algebra!

You Can Rework the Denominator

- Considering a low quality factor Q (roots are spread)

$$D(s) = 1 + \underbrace{b_1 s}_{\text{Low-frequency}} + \underbrace{b_2 s^2}_{\text{High-frequency}} = 1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2 \approx (1 + b_1 s)\left(1 + \frac{b_2}{b_1}s\right)$$

$$H(s) \approx \frac{1 + s r_C C_1}{\left[1 + s\left(C_1\left(r_C + R_1\right) + C_2 R_1\right)\right]\left(1 + s\frac{C_2 R_1 C_1 r_C}{C_1\left(r_C + R_1\right) + C_2 R_1}\right)}$$

$$H(s) = \frac{1 + \frac{s}{\omega_z}}{\left(1 + \frac{s}{\omega_{p_1}}\right)\left(1 + \frac{s}{\omega_{p_2}}\right)}$$

$$\omega_z = \frac{1}{r_C C_1} \qquad \omega_{p_2} = \frac{C_1\left(r_C + R_1\right) + C_2 R_1}{C_2 R_1 C_1 r_C}$$

$$\omega_{p_1} = \frac{1}{R_1\left(C_1 + C_2\right) + r_C C_1}$$

Check with Mathcad

- It is easy to check results versus a raw expression

$R_1 := 1k\Omega \quad r_C := 100\Omega \quad C_1 := 10nF \quad C_2 := 5nF \quad \|(x,y) := \frac{x \cdot y}{x+y}$

$H_0 := 1 \quad \tau_1 := C_1 \cdot (r_C + R_1) = 11\,\mu s \quad \tau_2 := C_2 \cdot R_1 = 5\,\mu s \quad a_1 := \tau_1 + \tau_2 = 16\,\mu s$

$\tau_{12} := C_2 \cdot (R_1 \| r_C) = 0.455\,\mu s \quad \tau_{21} := C_1 \cdot r_C \quad a_2 := \tau_2 \cdot \tau_{21} = 5\,\mu s^2$

$N_1(s) := 1 + s \cdot r_C \cdot C_1 \quad D_1(s) := 1 + a_1 \cdot s + a_2 \cdot s^2 \quad \boxed{H_1(s) := H_0 \cdot \frac{N_1(s)}{D_1(s)}}$

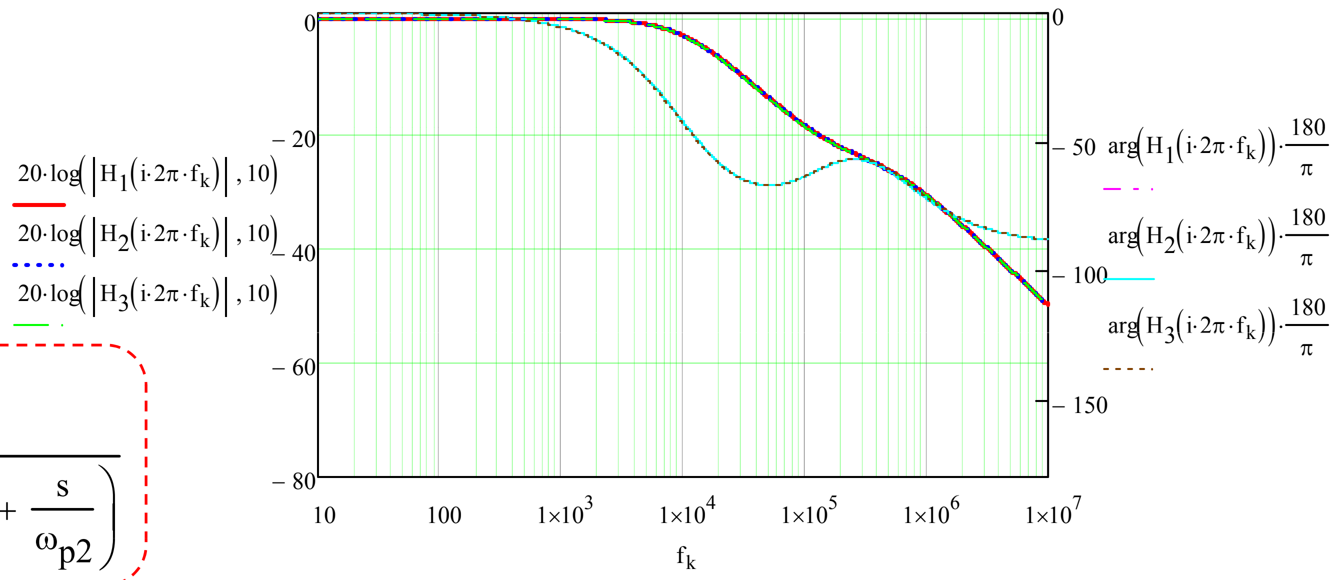
$$\boxed{\begin{aligned} Z_1(s) &:= \left(r_C + \frac{1}{s \cdot C_1}\right) \| \left(\frac{1}{s \cdot C_2}\right) \\ H_2(s) &:= \frac{Z_1(s)}{R_1 + Z_1(s)} \end{aligned}}$$

$$\omega_z := \frac{1}{r_C \cdot C_1}$$

$$\omega_{p1} := \frac{1}{a_1}$$

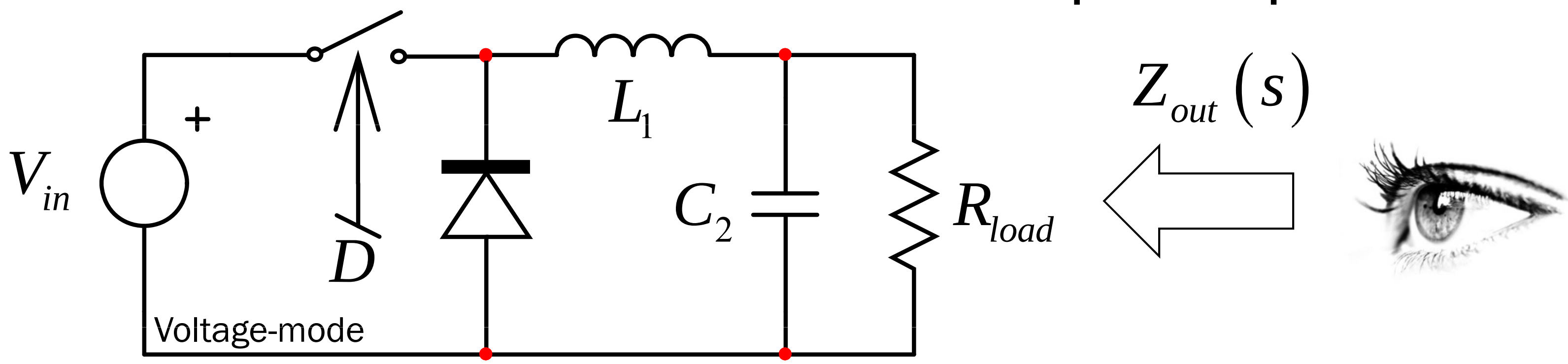
$$\omega_{p2} := \frac{a_1}{a_2}$$

$$\boxed{H_3(s) := \frac{1 + \frac{s}{\omega_z}}{\left(1 + \frac{s}{\omega_{p1}}\right) \cdot \left(1 + \frac{s}{\omega_{p2}}\right)}}$$

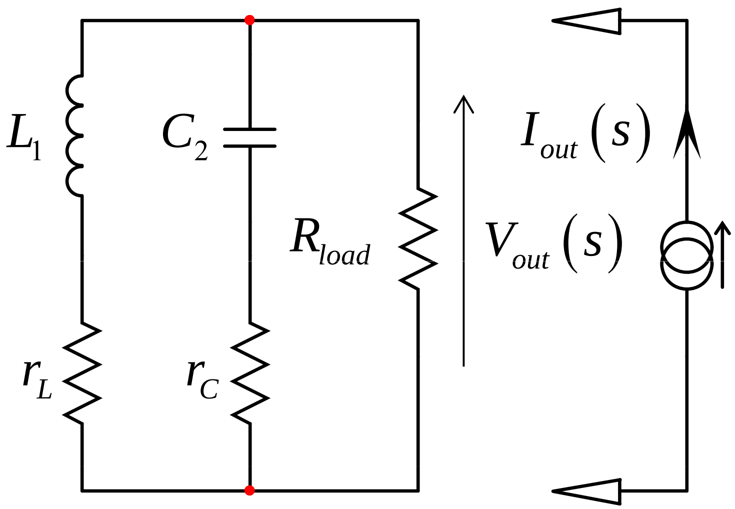


2nd-Order Example

- What is the buck converter output impedance?



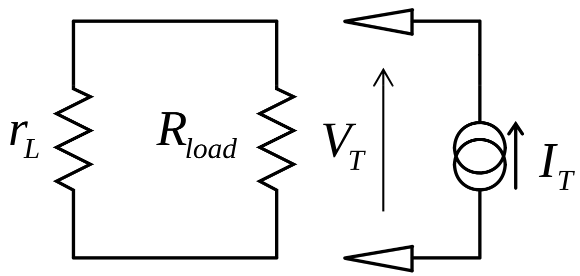
- Consider parasitic elements for L and C



$$Z_{out}(s) = \frac{V_{out}(s)}{I_{out}(s)} \begin{array}{l} \leftarrow \text{response} \\ \leftarrow \text{excitation} \end{array}$$

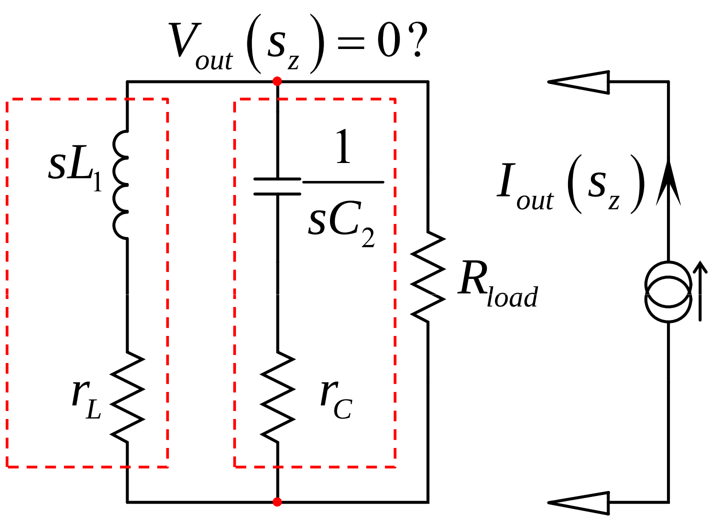
Buck Output Impedance

- Let's find the term R_0 in dc: open caps, short inductors



$$R_0 = \frac{V_T}{I_T} = r_L \parallel R_{load}$$

- The zeros cancel the response



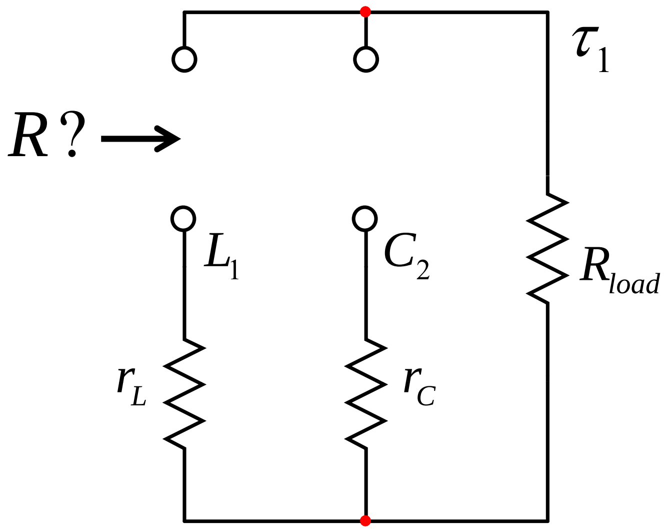
$$sL_1 + r_L = 0 \qquad r_L\left(s\frac{L_1}{r_L} + 1\right) = 0 \qquad \omega_{z_1} = \frac{r_L}{L_1}$$

$$\frac{1}{sC_2} + r_C = 0 \qquad 1 + sr_CC_2 = 0 \qquad \omega_{z_2} = \frac{1}{r_CC_2}$$

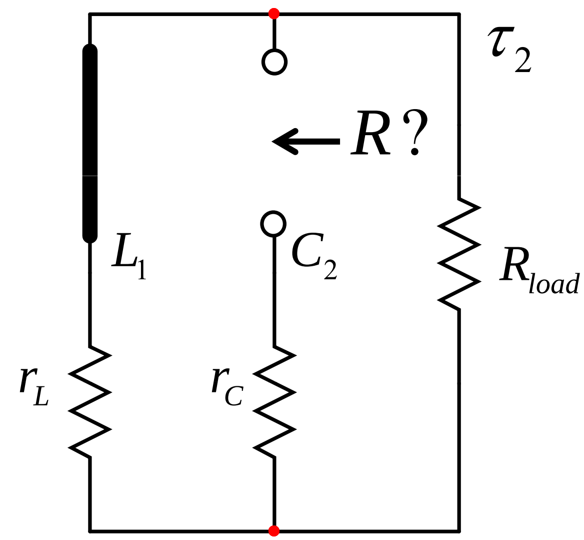
$$N(s) = \left(1 + s\frac{L_1}{r_L}\right)(1 + sr_CC_2)$$

Low-Frequency Time Constants

- All elements are in their dc state
  - Look at R driving L then R driving C



$$R = r_L + R_{load}$$

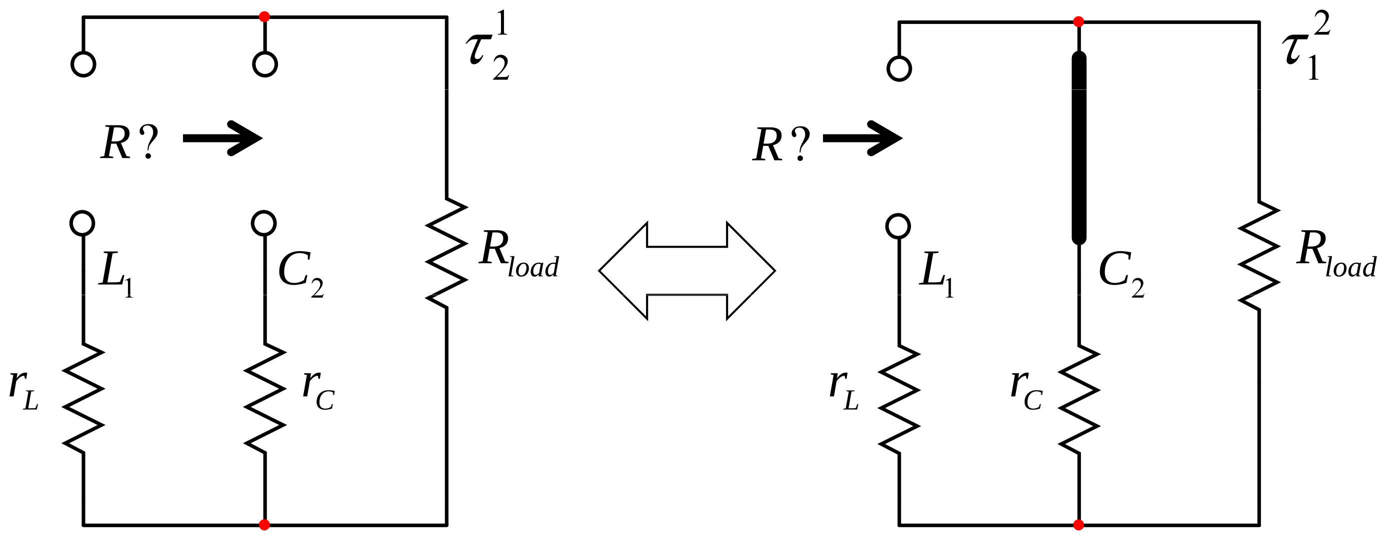


$$R = \left(r_L \parallel R_{load}\right) + r_C$$

$$b_1 = \frac{L_1}{r_L + R_{load}} + C_2\left[\left(r_L \parallel R_{load}\right) + r_C\right]$$

High-Frequency Time Constants

- Set L_1 in high frequency state and look at R driving C_2



$$\tau_2^1 = C_2\left(r_c + R_{load}\right)$$

$$\tau_1^2 = \frac{L_1}{r_L + R_{load} \parallel r_C}$$

$$b_2 = \frac{L_1}{r_L + R_{load}} C_2\left(r_c + R_{load}\right) = C_2\left[\left(r_L \parallel R_{load}\right) + r_C\right]\frac{L_1}{r_L + R_{load} \parallel r_C}$$

$$b_2 = \tau_1 \tau_2^1 \qquad b_2 = \tau_2 \tau_1^2$$

Compensating the Buck – Method 1

- We have our denominator!

$$D(s) = 1 + s\left(\frac{L_1}{r_L + R_{load}} + C_2\left[\left(r_L \parallel R_{load}\right) + r_C\right]\right) + s^2\left(L_1 C_2 \frac{r_C + R_{load}}{r_L + R_{load}}\right)$$

- The complete transfer function is now:

$$Z_{out}(s) = \left(r_L \parallel R_{load}\right)\frac{\left(1 + s\frac{L_1}{r_L}\right)\left(1 + s r_C C_2\right)}{1 + s\left(\frac{L_1}{r_L + R_{load}} + C_2\left[\left(r_L \parallel R_{load}\right) + r_C\right]\right) + s^2\left(L_1 C_2 \frac{r_C + R_{load}}{r_L + R_{load}}\right)}$$

Compensating the Buck – Method 1

- It can be put under the following form:

$$Z_{out}(s) = R_0 \frac{\left(1 + s/\omega_{z_1}\right)\left(1 + s/\omega_{z_2}\right)}{1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2}$$

- We can identify the terms:

$$R_0 = r_L \,||\, R_{load} \qquad \omega_{z_1} = \frac{r_L}{L_1} \qquad \omega_{z_2} = \frac{1}{r_C C_2}$$

$$\omega_0 = \frac{1}{\sqrt{L_1 C_2}} \sqrt{\frac{r_L + R_{load}}{r_C + R_{load}}} \qquad Q = \frac{L_1 C_2 \omega_0 \left(r_C + R_{load}\right)}{L_1 + C_2\left(r_L r_C + r_L R_{load} + r_C R_{load}\right)}$$

Check Response with Mathcad®

- Express all time constants independently

$r_L := 0.1\Omega \quad r_C := 10\Omega \quad C_2 := 10nF \quad L_1 := 20\mu H \quad R_L := 5\Omega \quad \|(x,y) := \frac{x \cdot y}{x+y}$

$R_0 := r_L \| R_L = 0.098\Omega$

$\tau_1 := \frac{L_1}{r_L + R_L} = 3.922\mu s \quad \tau_2 := C_2 \cdot (r_L \| R_L + r_C) = 100.98ns \quad a_1 := \tau_1 + \tau_2 = 4.023\mu s$

$\tau_{12} := C_2 \cdot (r_C + R_L) = 0.15\mu s \quad \tau_{21} := \frac{L_1}{r_L + R_L \| r_C} = 5.825\mu s \quad a_2 := \tau_2 \cdot \tau_{21} = 0.588\mu s^2$

$N_1(s) := \left(1 + s \cdot \frac{L_1}{r_L}\right) \cdot (1 + s \cdot r_C \cdot C_2) \quad D_1(s) := 1 + a_1 \cdot s + a_2 \cdot s^2 \quad Z_1(s) := R_0 \cdot \frac{N_1(s)}{D_1(s)}$

$\omega_{z1} := \frac{r_L}{L_1} \quad \omega_{z2} := \frac{1}{r_C \cdot C_2} \quad \omega_0 := \frac{1}{\sqrt{L_1 \cdot C_2}} \cdot \sqrt{\frac{r_L + R_L}{r_C + R_L}} \quad Q := \frac{L_1 \cdot C_2 \cdot \omega_0 \cdot (r_C + R_L)}{L_1 + C_2 \cdot (r_L \cdot r_C + r_L \cdot R_L + r_C \cdot R_L)}$

$$Z_3(s) := R_0 \cdot \frac{\left(1 + \frac{s}{\omega_{z1}}\right) \cdot \left(1 + \frac{s}{\omega_{z2}}\right)}{1 + \frac{s}{\omega_0 \cdot Q} + \left(\frac{s}{\omega_0}\right)^2}$$

$Z_a(s) := s \cdot L_1 + r_L \quad Z_b(s) := \frac{1}{s \cdot C_2} + r_C$

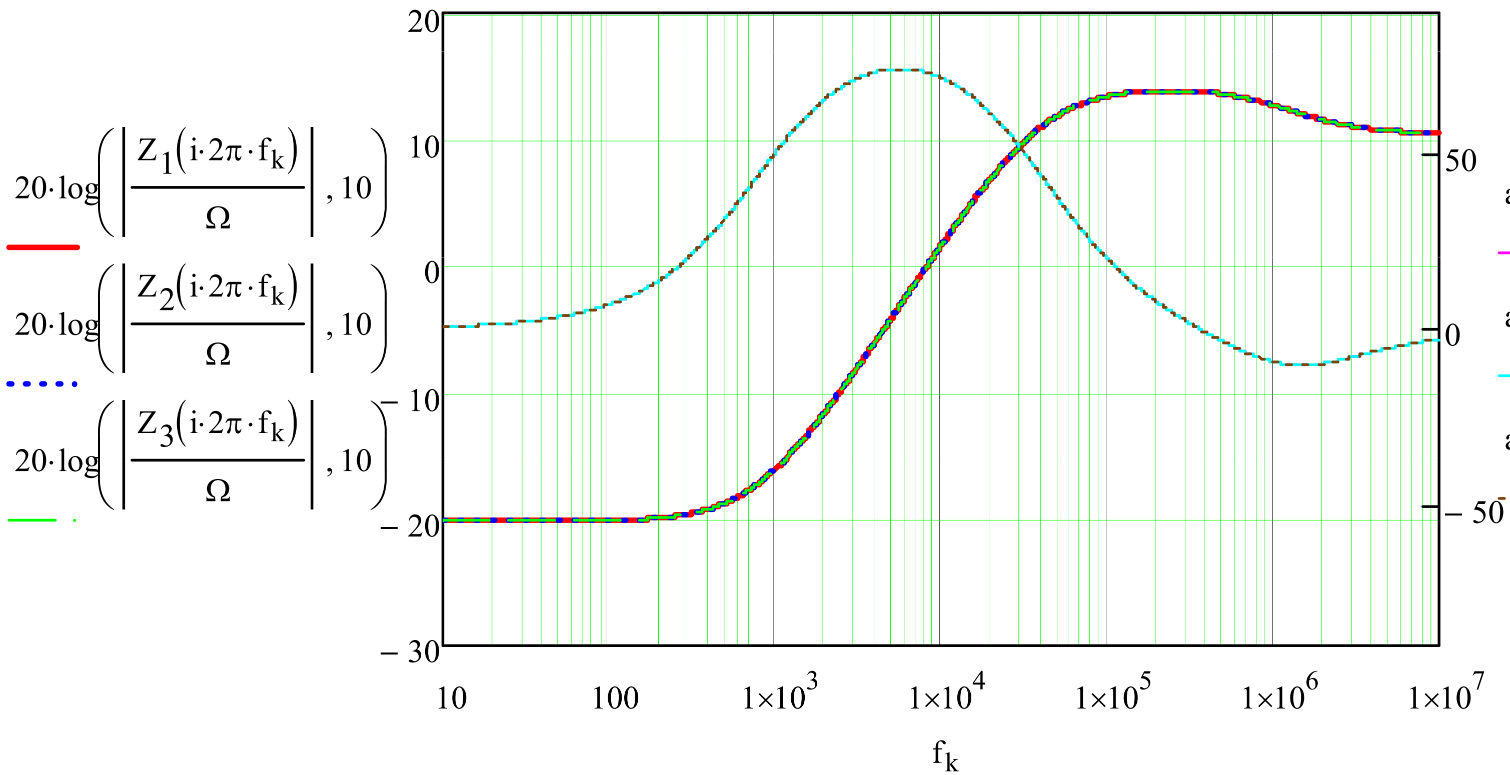
$Z_2(s) := R_L \| (Z_a(s) \| Z_b(s))$

Raw expression

✓Fault correction is easy!

Derivation is Correct

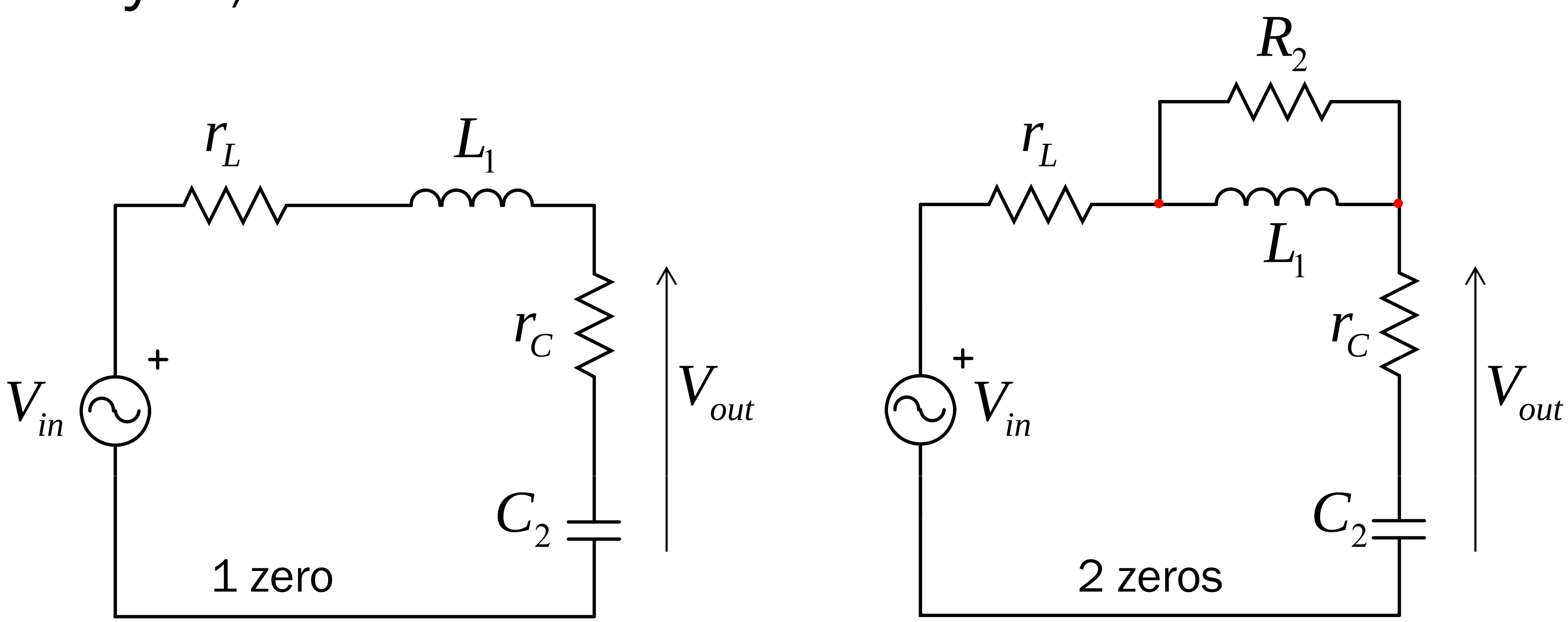
- Magnitude and phase curves perfectly superimpose



- Always verify results with a different expression or SPICE

Checking for Zeros

- ❑ Is there a quick way to check if there are zeros?
- ➢ Yes! Simultaneously put reactances in their HF state
- ➢ Check if the response is still there
- ✓ If yes, there are 2 zeros in the circuit

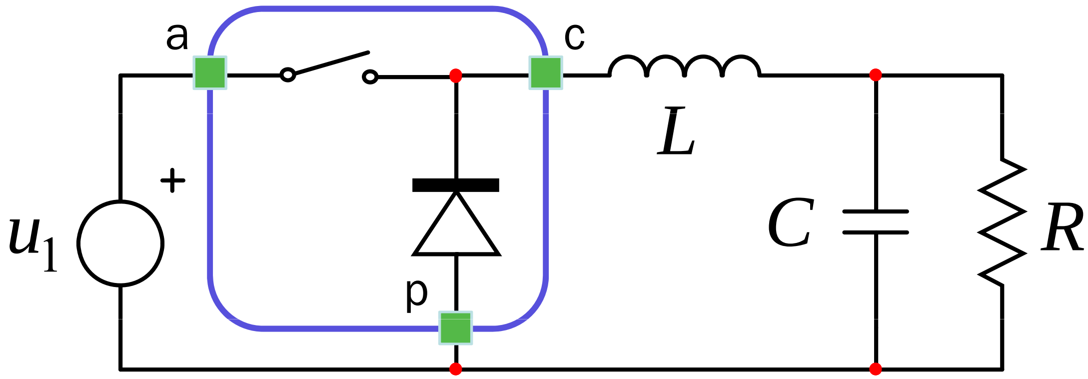


Course Agenda

- ❑ What is a Transfer Function?
- ❑ Why do We Need New Analytical Techniques?
- ❑ Time Constants and Poles
- ❑ Identifying the Zeros
- ❑ The Null Double Injection
- ❑ 2nd-Order Networks
- ❑ **The PWM Switch Model**
- ❑ A CCM Buck in Voltage Mode
- ❑ A CCM Buck-Boost in Voltage Mode

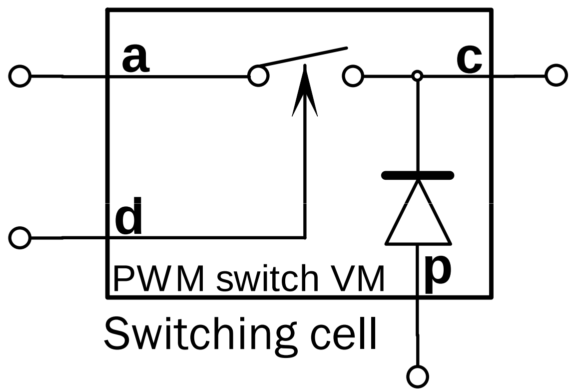
The PWM Switch Model in Voltage Mode

- The non-linearity is brought by the switching cell

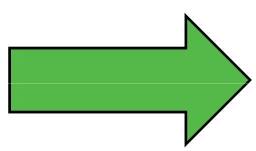


a: active
c: common
p: passive

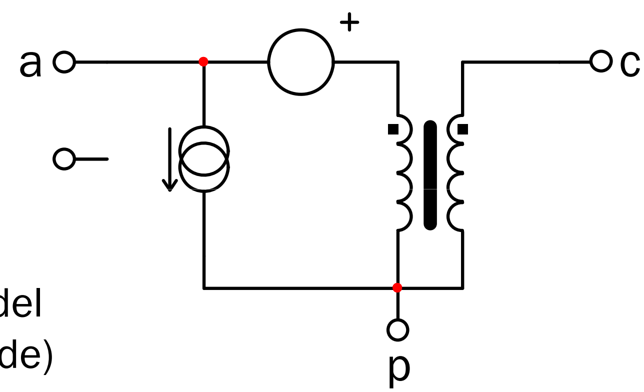
- Why don't we linearize the cell alone?



Switching cell



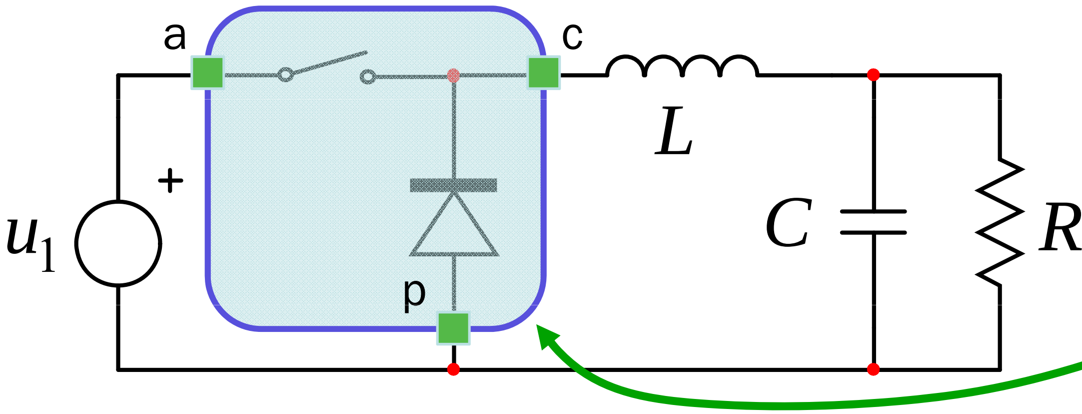
Small-signal model
(CCM voltage-mode)

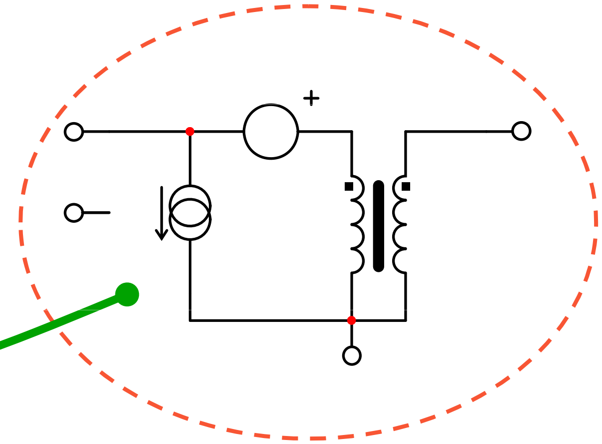


V. Vorpérian, "Simplified Analysis of PWM Converters using Model of PWM Switch, parts I and II" IEEE Transactions on Aerospace and Electronic Systems, Vol. 26, NO. 3, 1990

Replace the Switches by the Model

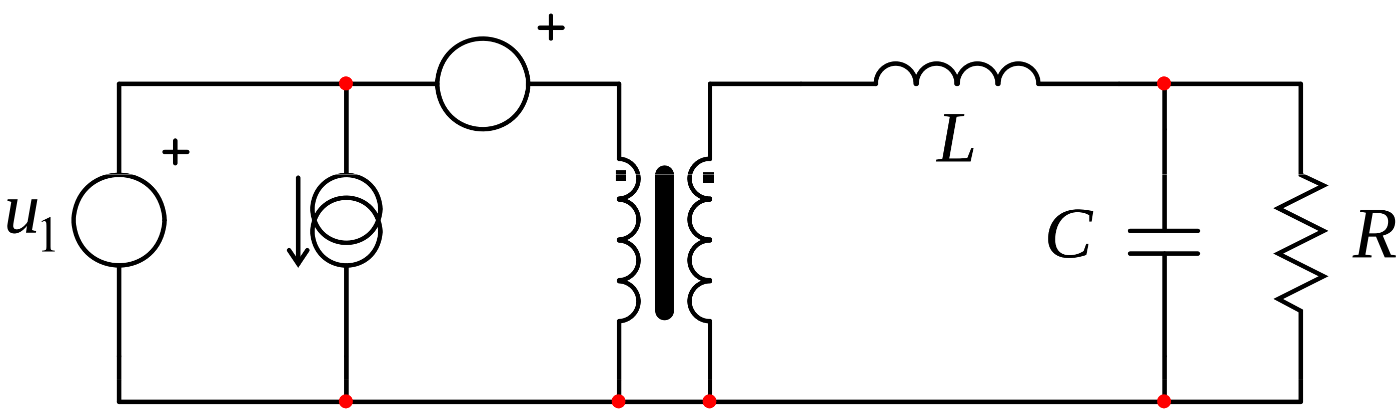
- ❑ Like in a bipolar circuit, replace the switching cell...





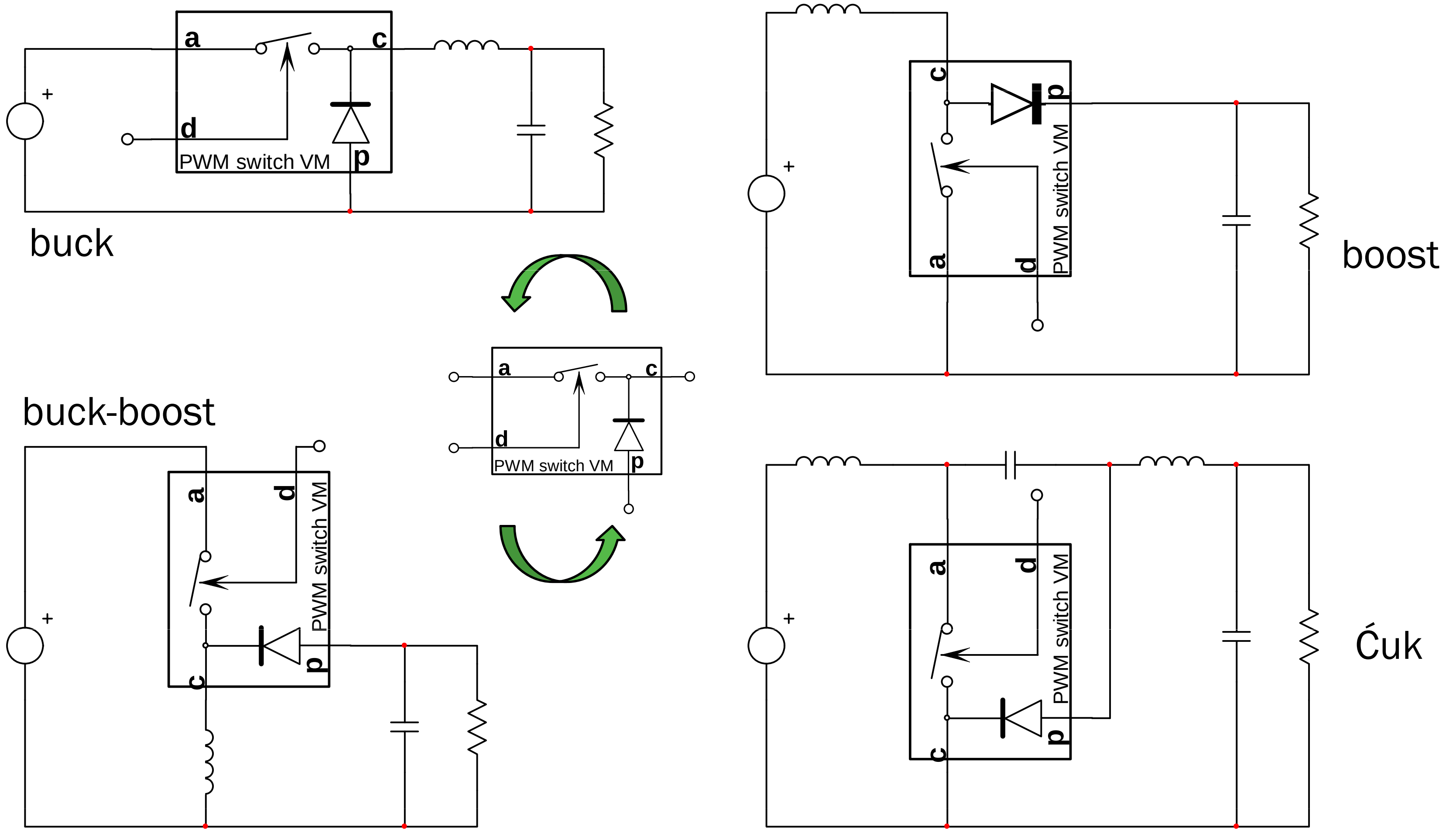
Small-signal model

- ❑ ...and solve a set of linear equations!



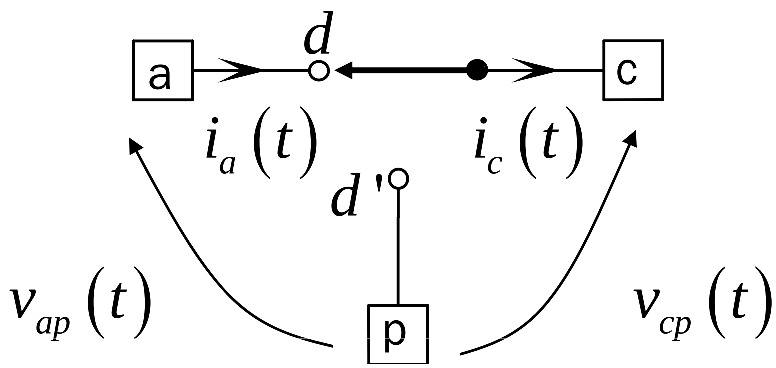
An Invariant Model

- The switching cell made of two switches is everywhere!

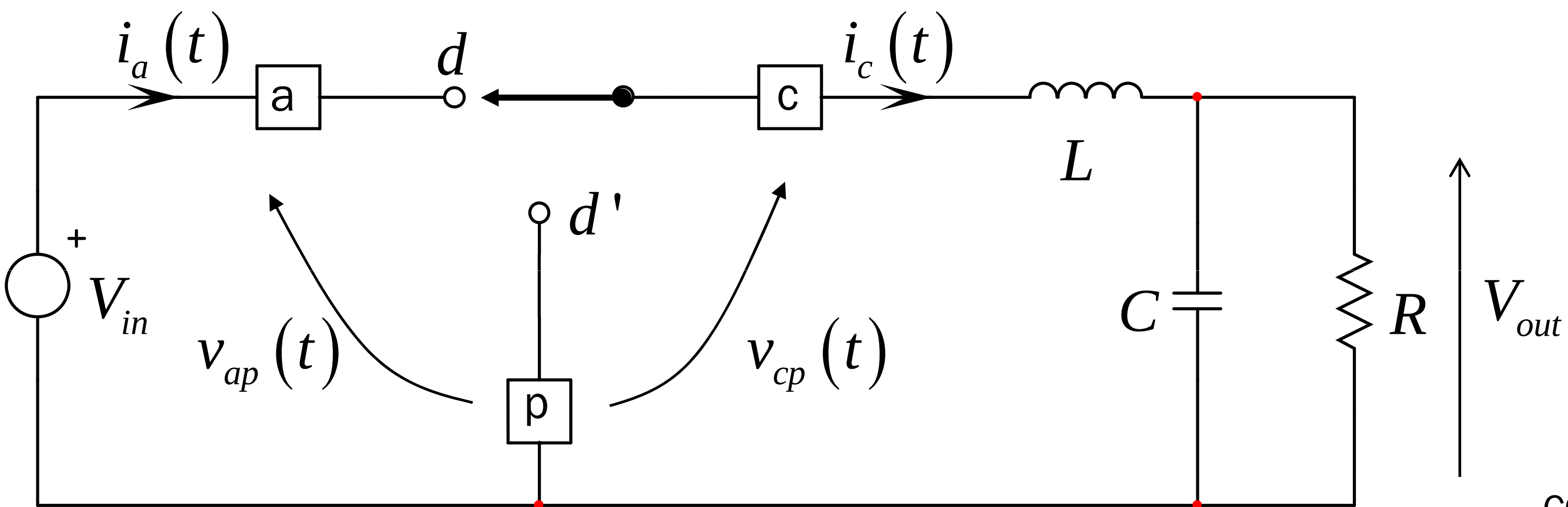


CCM Common Passive Configuration

- The PWM switch is a single-pole double-throw model



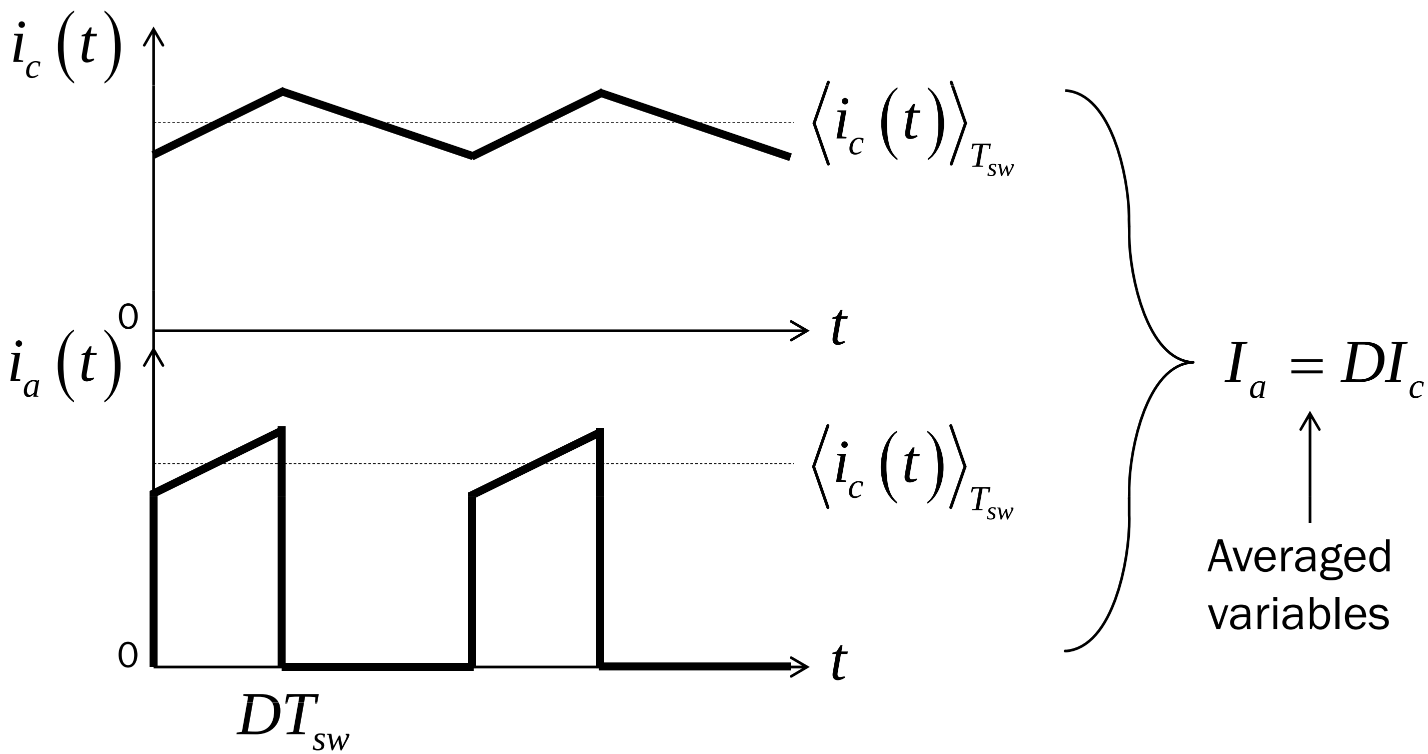
- Install it in a buck and draw its terminals waveforms



CCM VM

The Common Passive Configuration

- ❑ Average the current waveforms across the PWM switch

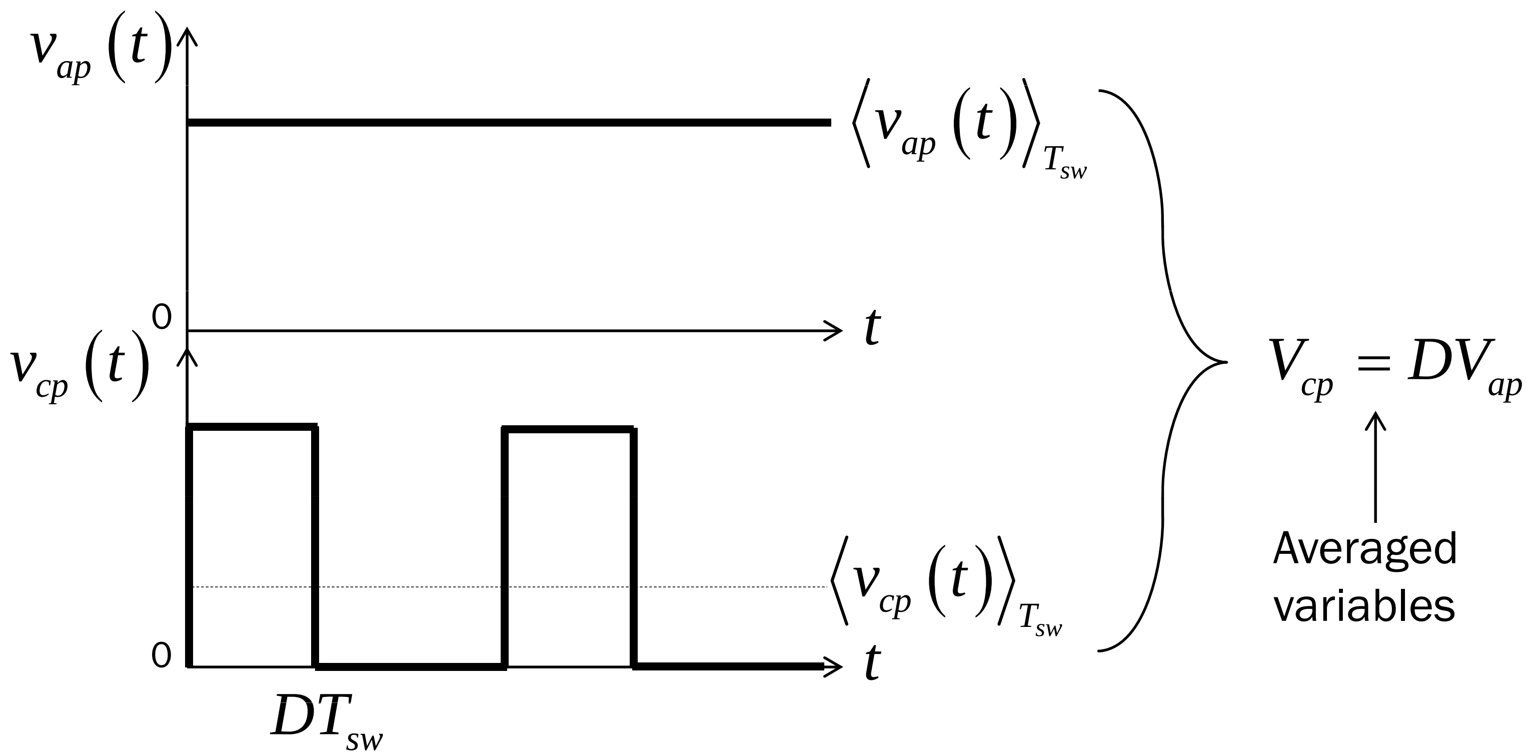


$$\langle i_a(t)\rangle_{T_{sw}} = I_a = \frac{1}{T_{sw}}\int_0^{DT_{sw}} i_a(t)\,dt = D\langle i_c(t)\rangle_{T_{sw}} = DI_c$$

CCM VM

The Common Passive Configuration

- Average the voltage waveforms across the PWM switch

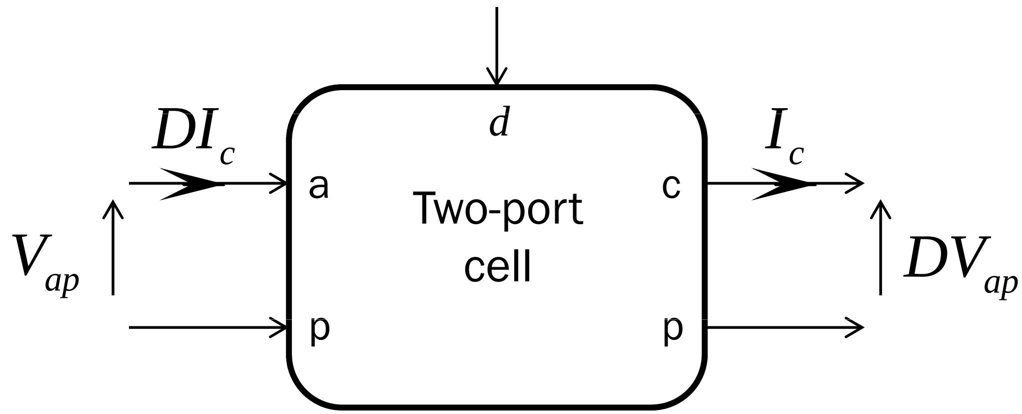


$$\langle v_{cp}(t) \rangle_{T_{sw}} = V_{cp} = \frac{1}{T_{sw}} \int_{0}^{DT_{sw}} v_{cp}(t)\,dt = D \langle v_{ap}(t) \rangle_{T_{sw}} = DV_{ap}$$

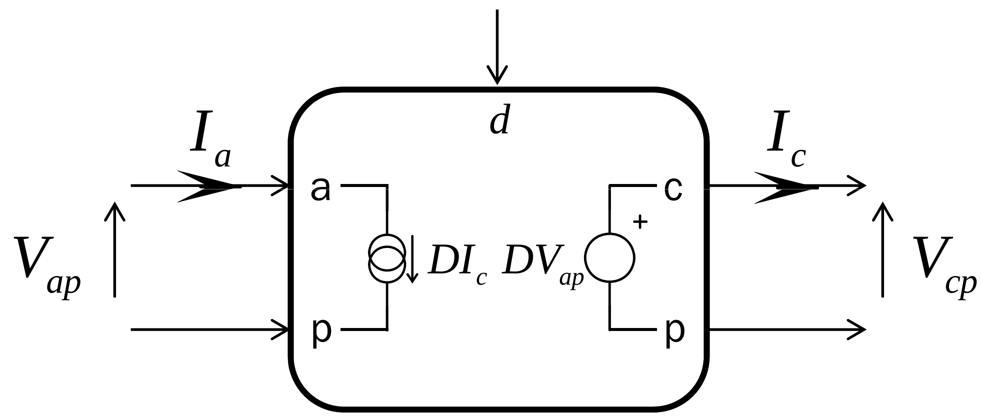
CCM VM

A Two-Port Representation

- We have a link between input and output variables



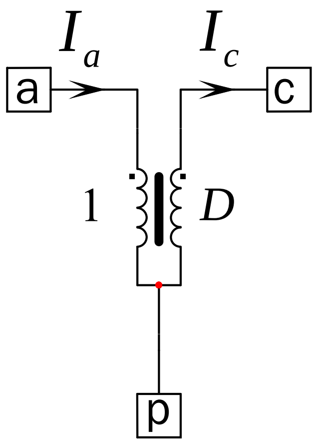
- We can involve current and voltage sources



CCM VM

A Dc Transformer Model

- The large-signal model is a dc "transformer"!

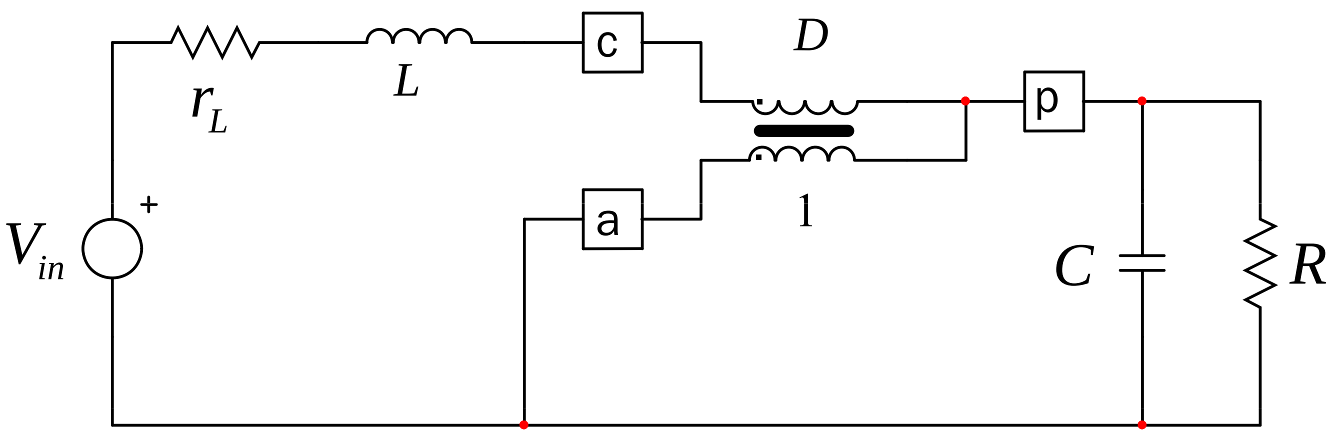


$$I_a = DI_c \qquad I_c = \frac{I_a}{D}$$

$$V_{ap} = \frac{V_{cp}}{D} \qquad V_{cp} = DV_{ap}$$

dc equations!

- It can be plugged into any 2-switch CCM converter

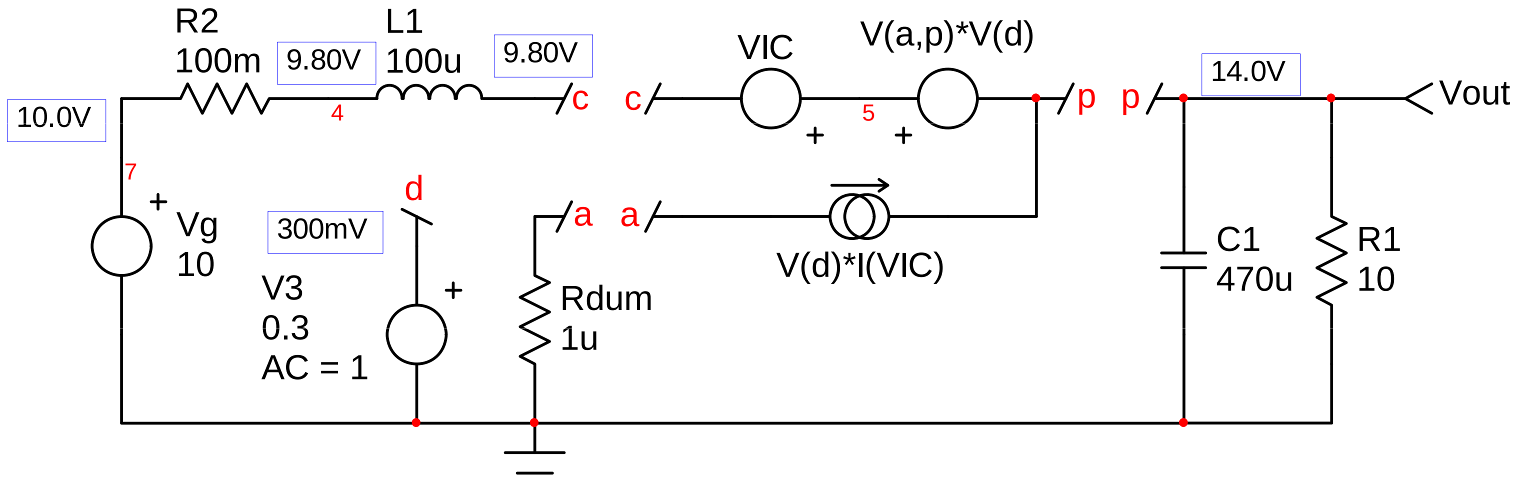


Dc bias point
Ac response

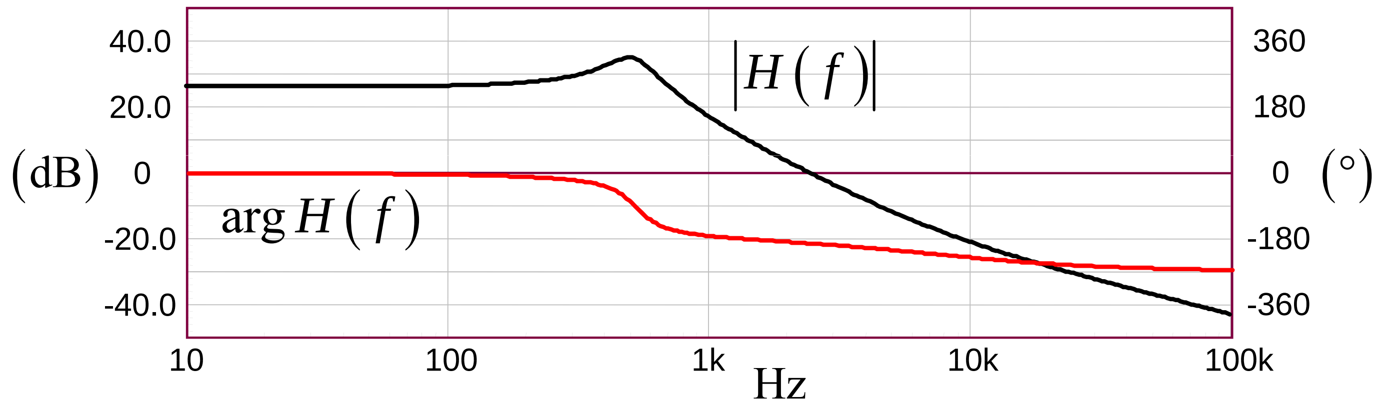
CCM VM

Simulate Immediately!

- SPICE can get you the dc bias point



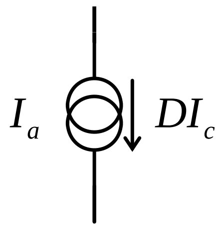
- ...but also the ac response as it linearizes the circuit



CCM

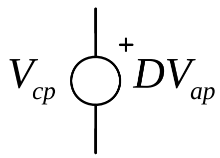
A Small-Signal Model

- We need a small-signal version to get the ac response
  - Perturb equations or run partial differentiation



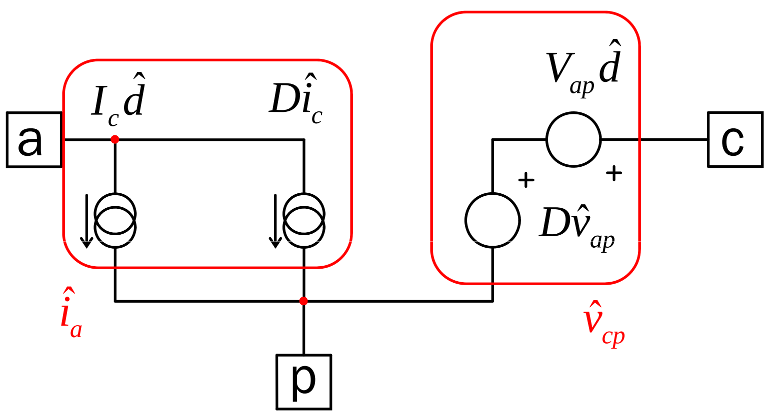
2 variables

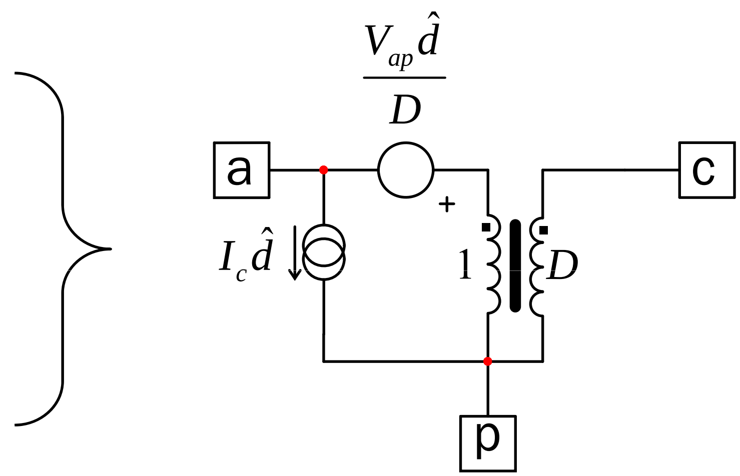
$$\hat{i}_a = \frac{\partial f\left(D, I_c\right)}{\partial D}\hat{d} + \frac{\partial f\left(D, I_c\right)}{\partial I_c}\hat{i}_c \quad \Rightarrow \quad \hat{i}_a = I_c\hat{d} + D\hat{i}_c$$



2 variables

$$\hat{v}_{cp} = \frac{\partial f\left(D, V_{ap}\right)}{\partial D}\hat{d} + \frac{\partial f\left(D, V_{ap}\right)}{\partial V_{ap}}\hat{v}_{ap} \quad \Rightarrow \quad \hat{v}_{cp} = V_{ap}\hat{d} + D\hat{v}_{ap}$$





Small-signal model

CCM

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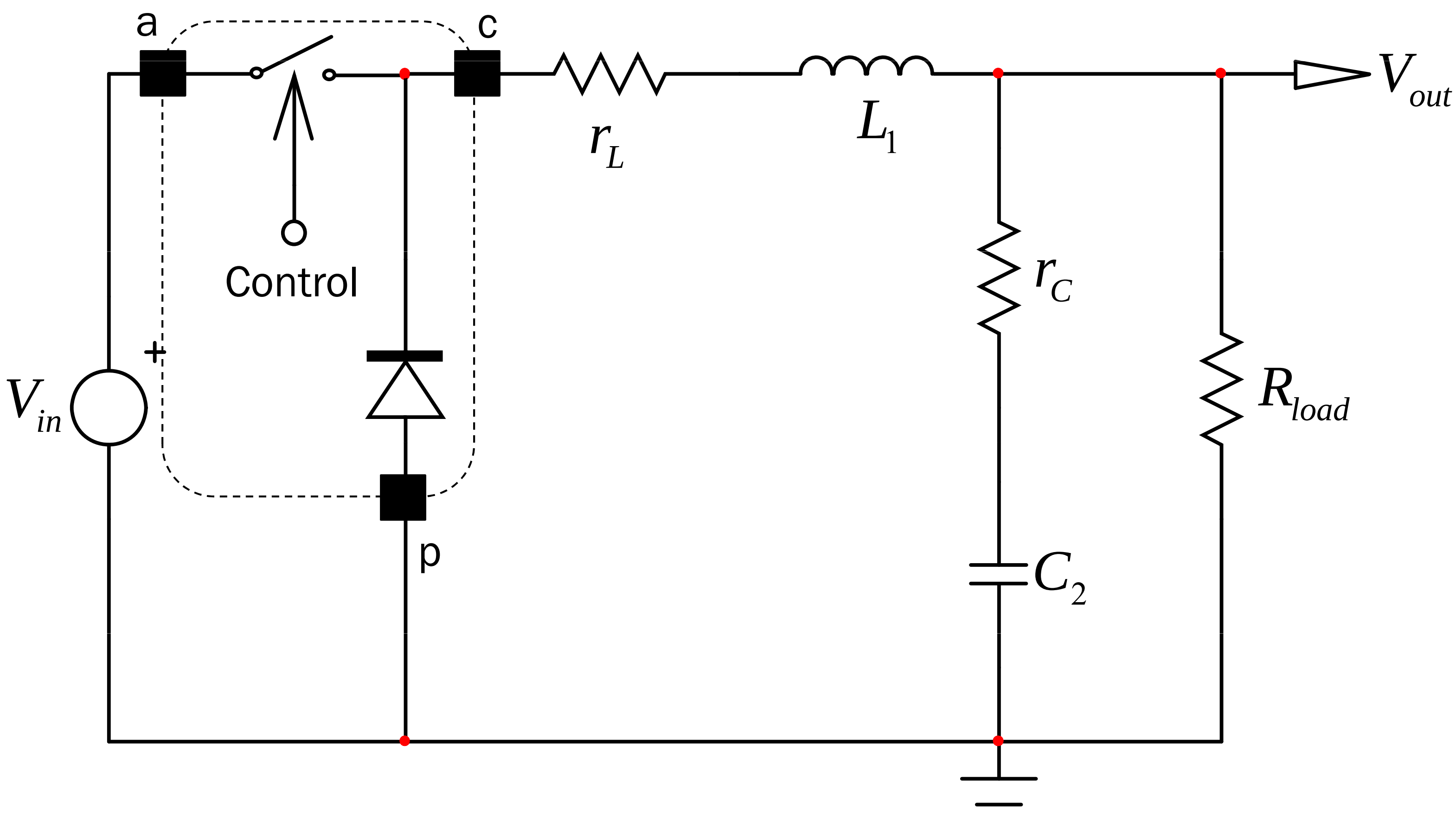
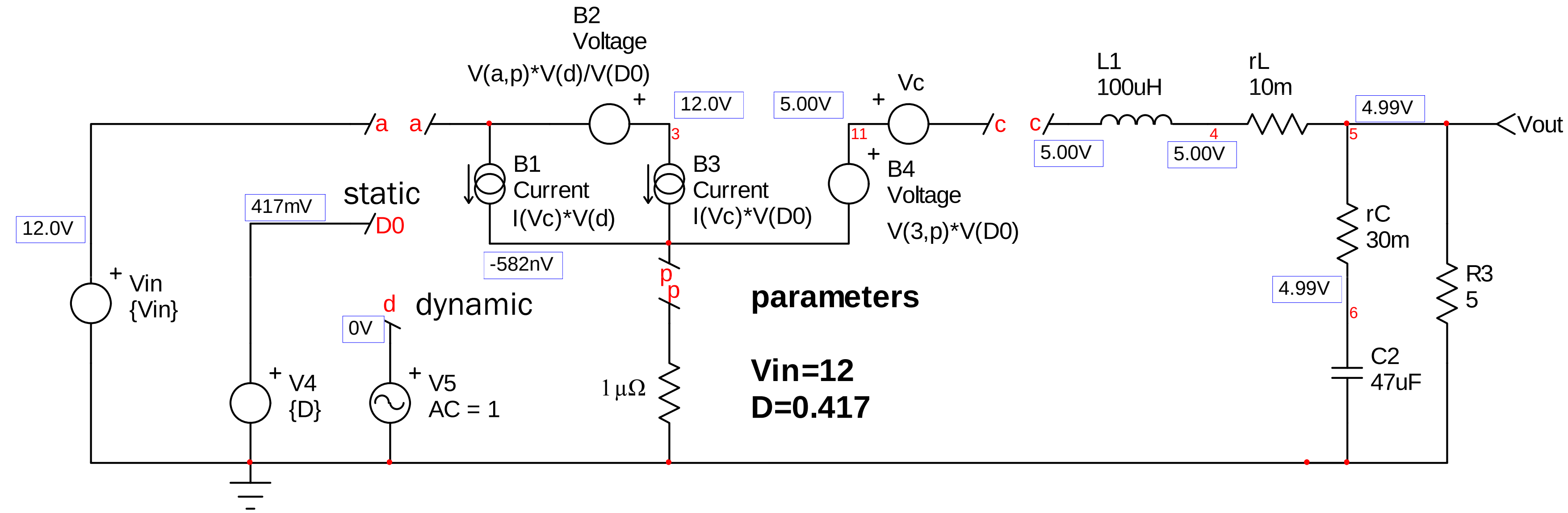
A Buck Converter

- Replace the diode and the switch by the model



Model at Work in a Buck Converter

- Plug the invariant small-signal model: all linear!



- We want the ac control-to-output transfer function

$$\left.\frac{V_{out}(s)}{D(s)}\right|_{\hat{v}_{in}=0}$$

Set node *a* to 0 V

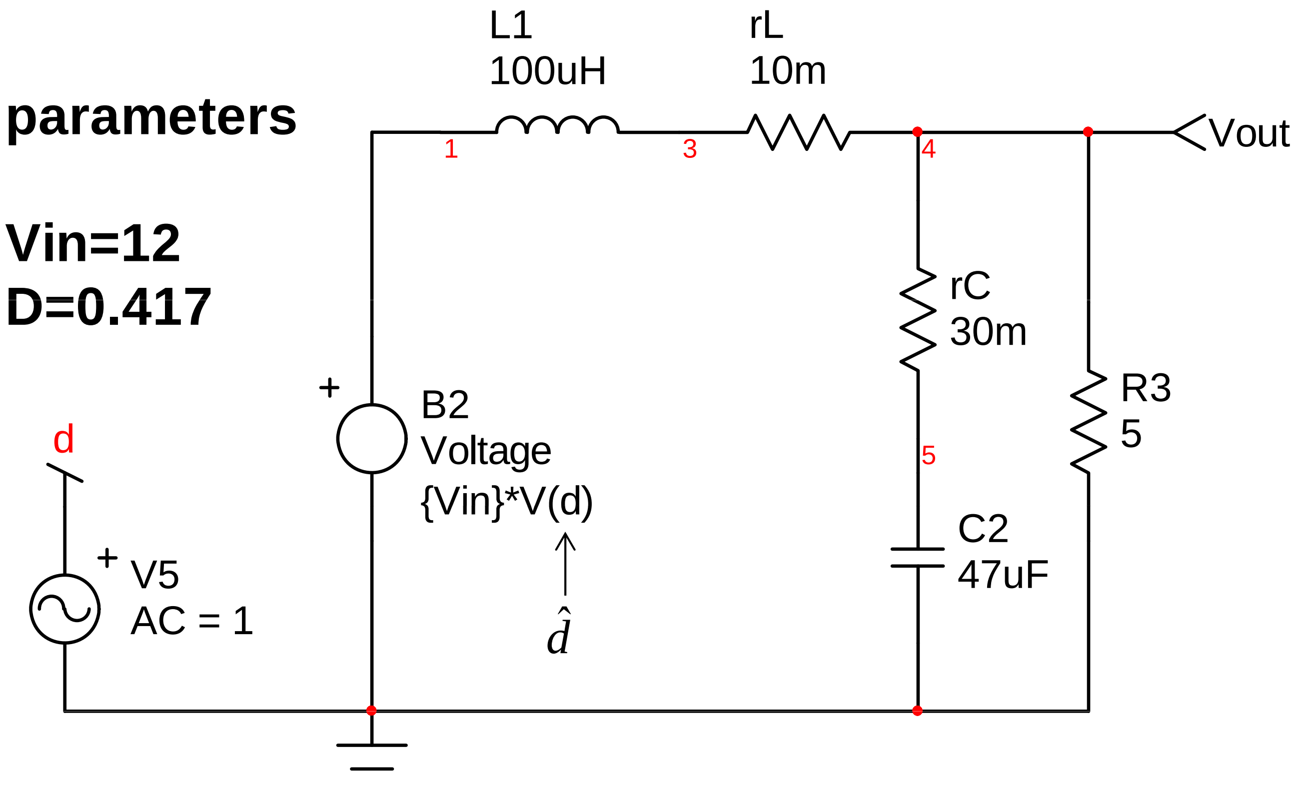
Node *p* is ground

$$V(a,p)=V_{in}$$

Simplify schematic

Redraw the Simplified Circuit

- Ac contribution from the input is not the subject

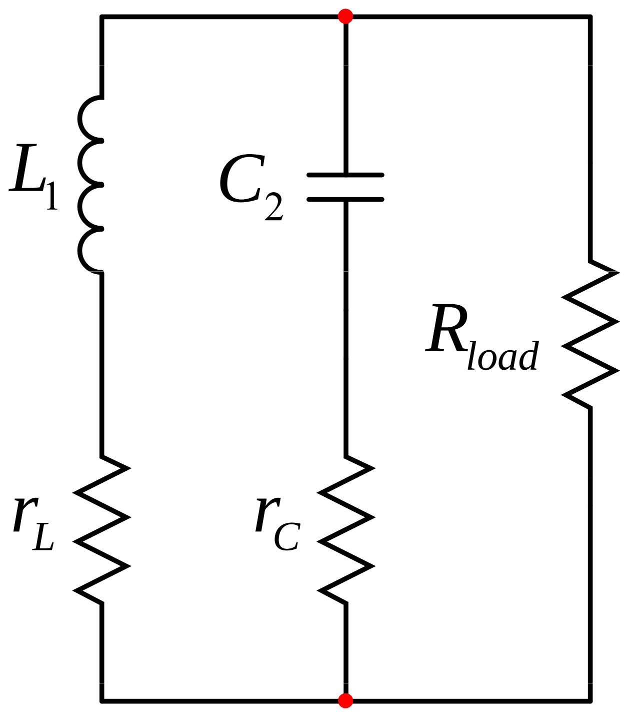


- Setting d to 0 V turns the excitation off

Control-to-output

A Familiar Architecture

- The circuit returns to its natural structure



$$D(s) = 1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2$$

$$\omega_0 = \frac{1}{\sqrt{L_1 C_2}} \sqrt{\frac{r_L + R_{load}}{r_C + R_{load}}}$$

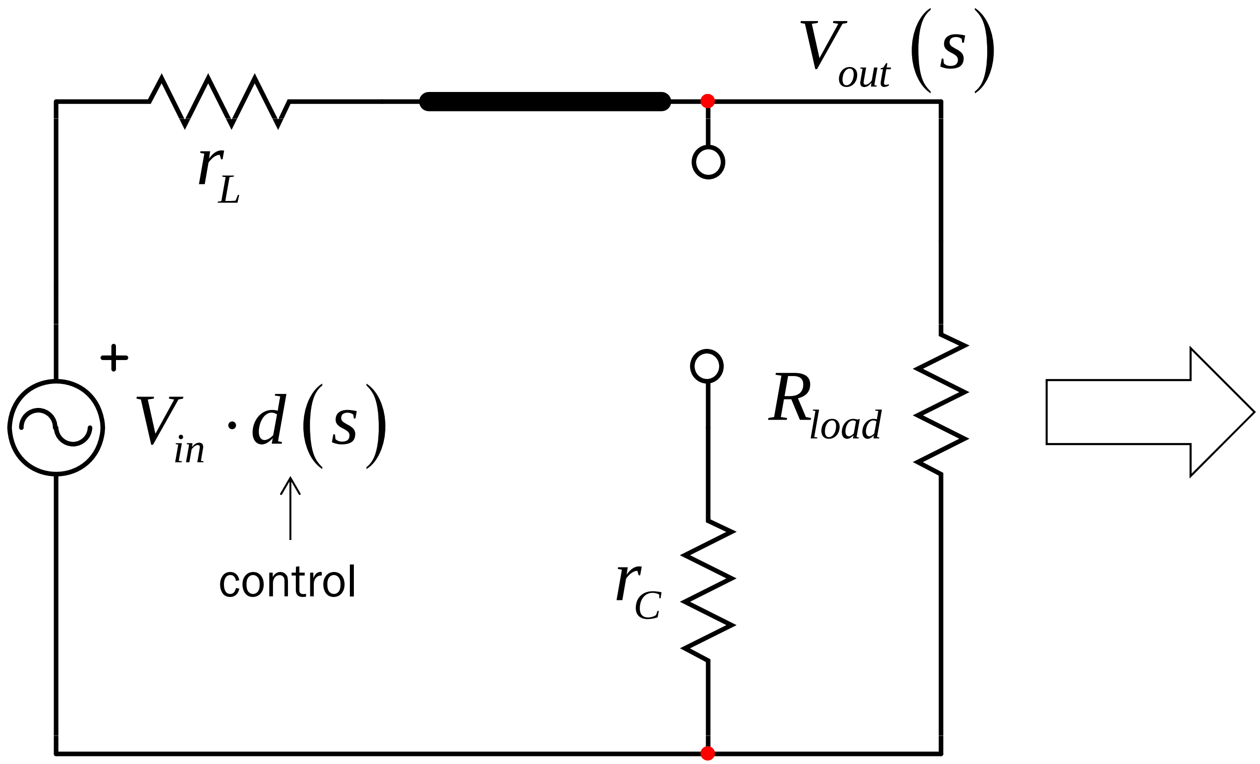
$$Q = \frac{L_1 C_2 \omega_0 \left(r_C + R_{load}\right)}{L_1 + C_2 \left(r_L r_C + r_L R_{load} + r_C R_{load}\right)}$$

- You can reuse the denominator previously determined

Control-to-output

Determine the Gain in Dc

- Open the capacitor, short the inductor



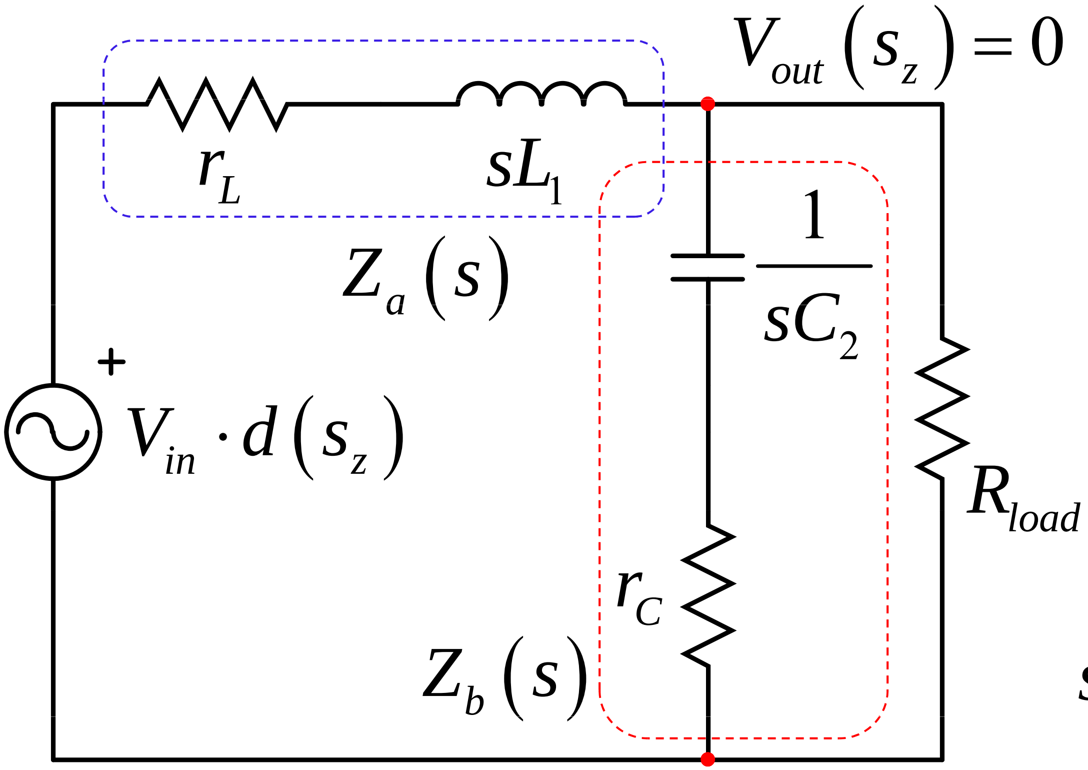
$$H_0 = V_{in} \frac{R_{load}}{R_{load} + r_L}$$

- Losses from the MOSFET and the diode could be added

Control-to-output

Determining the Zeros

- The response is canceled if $Z_2(s_z)$ is a transformed short



$$sL_1 + r_L \rightarrow \infty \quad \text{Non !}$$

$$r_C + \frac{1}{sC_2} = \frac{sr_C C_2 + 1}{sC_2} = 0\,?$$

Oui !

$$s_z = -\frac{1}{r_C C_2} \qquad \omega_z = \frac{1}{r_C C_2}$$

- Observe the transformed network at $s = s_z$

Control-to-output

Final Response

- Assemble the pieces to form $H(s)$

$$\frac{V_{out}(s)}{D(s)} = H_0 \frac{1+\frac{s}{\omega_z}}{1+\frac{s}{\omega_0 Q}+\left(\frac{s}{\omega_0}\right)^2} \qquad \omega_z = \frac{1}{r_C C_2} \qquad H_0 = V_{in}\frac{R_{load}}{R_{load}+r_L}$$

$$\omega_0 = \frac{1}{\sqrt{L_1 C_2}}\sqrt{\frac{r_L+R_{load}}{r_C+R_{load}}} \qquad Q = \frac{L_1 C_2 \omega_0 \left(r_C + R_{load}\right)}{L_1 + C_2\left(r_L r_C + r_L R_{load} + r_C R_{load}\right)}$$

- Compare low-entropy form with raw formula

$$H_{ref}(s) = V_{in}\frac{R_L \,||\, Z_b(s)}{Z_a(s) + R_L \,||\, Z_b(s)}$$

Control-to-output

Test with Mathcad is Simple and Fast

$r_L := 0.01\Omega$ $\quad r_C := 0.03\Omega$ $\quad C_2 := 47\mu F$ $\quad L_1 := 100\mu H$ $\quad R_L := 5\Omega$ $\quad \|(x,y) := \frac{x \cdot y}{x+y}$

$Z_a(s) := s \cdot L_1 + r_L$ $\quad Z_b(s) := \frac{1}{s \cdot C_2} + r_C$

$V_{in} := 12V$ $\quad V_p := 1V$ $\quad H_0 := \frac{V_{in}}{V_p} \cdot \frac{R_L}{R_L + r_L} = 11.976$ $\quad 20 \cdot \log(H_0) = 21.566$

$H_3(s) := \frac{Z_b(s) \| R_L}{Z_a(s) + Z_b(s) \| R_L} \cdot \frac{V_{in}}{V_p}$

$\tau_1 := \frac{L_1}{r_L + R_L} = 19.96\,\mu s$ $\quad \tau_2 := C_2 \cdot (r_L \| R_L + r_C) = 1.879 \times 10^3 \cdot ns$

$a_1 := \tau_1 + \tau_2 = 21.839\,\mu s$

$\tau_{12} := C_2 \cdot (r_C + R_L) = 236.41 \cdot \mu s$ $\quad \tau_{21} := \frac{L_1}{r_L + R_L \| r_C} = 2.511 \times 10^3 \cdot \mu s$

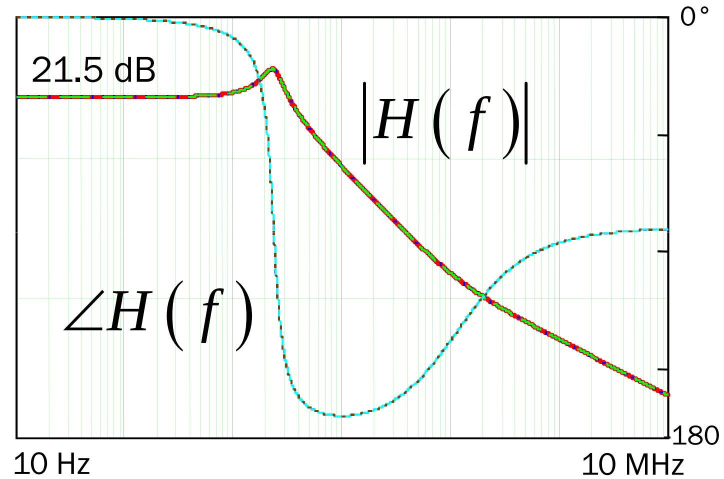
$a_2 := \tau_1 \cdot \tau_{12} = 4.719 \times 10^3 \cdot \mu s^2$

$N_1(s) := 1 + s \cdot r_C \cdot C_2$ $\quad D_1(s) := 1 + a_1 \cdot s + a_2 \cdot s^2$ $\quad H_1(s) := H_0 \cdot \frac{N_1(s)}{D_1(s)}$

$\omega_{z2} := \frac{1}{r_C \cdot C_2}$ $\quad \omega_0 := \frac{1}{\sqrt{L_1 \cdot C_2}} \cdot \sqrt{\frac{r_L + R_L}{r_C + R_L}}$ $\quad Q := \frac{L_1 \cdot C_2 \cdot \omega_0 \cdot (r_C + R_L)}{L_1 + C_2 \cdot (r_L \cdot r_C + r_L \cdot R_L + r_C \cdot R_L)}$

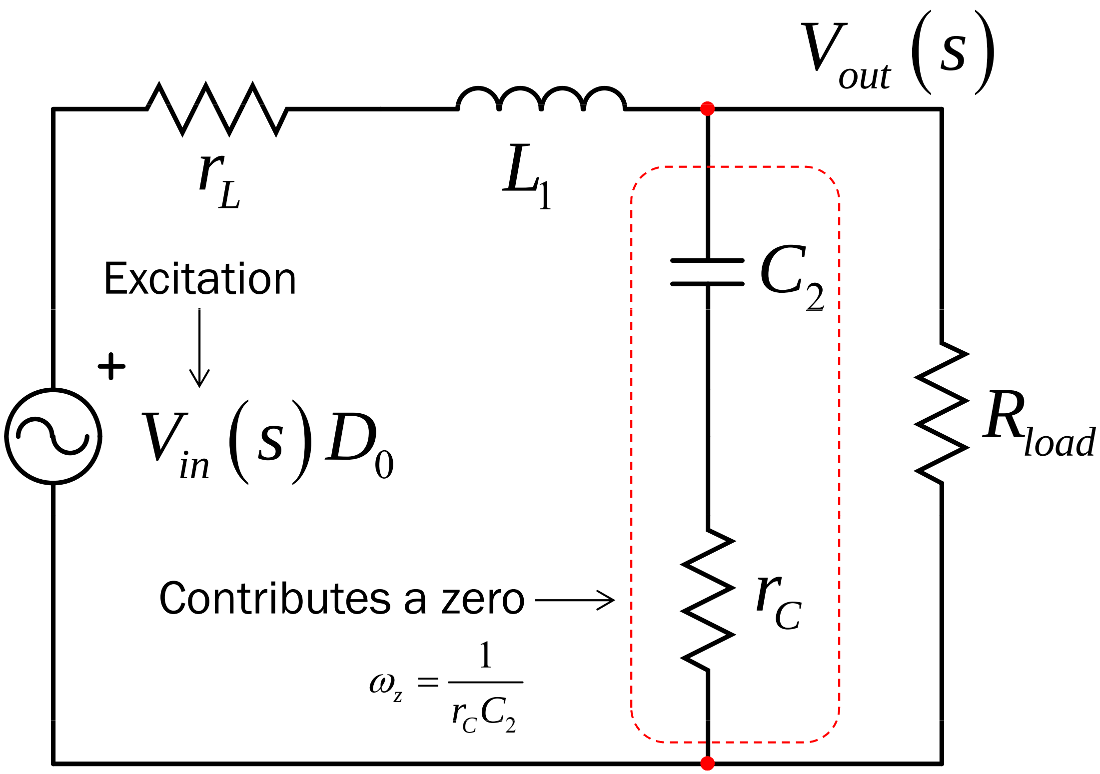
$$H_2(s) := H_0 \cdot \frac{1 + \frac{s}{\omega_{z2}}}{1 + \frac{s}{\omega_0 \cdot Q} + \left(\frac{s}{\omega_0}\right)^2}$$

All curves superimpose!



Input to Output Transfer Function

- We can set $\hat{d}$ to 0 and check V_{out} to V_{in}



Set L_1 as a short circuit
open capacitor C_2

$$H_0 = D_0 \frac{R_{load}}{R_{load} + r_L}$$

- For $V_{in} = 0$, same structure as before, reuse $D(s)$!

Transfer Function is Immediate

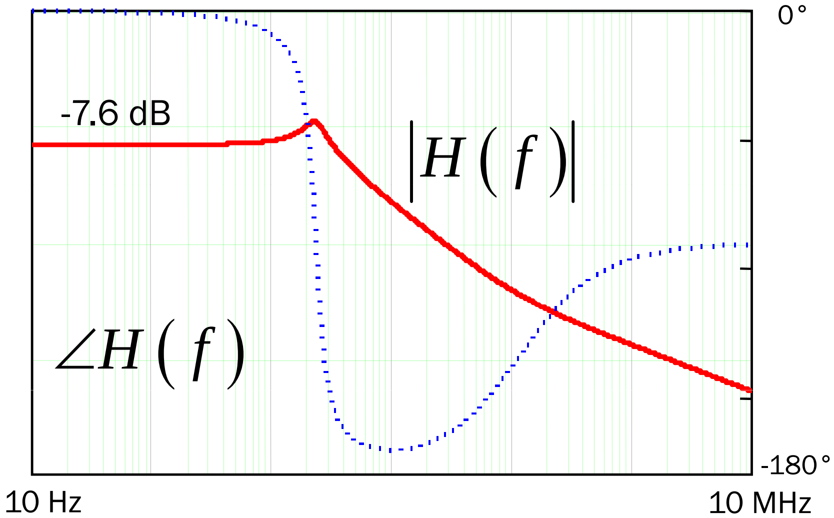
- Reuse existing formula and build transfer function

$$\frac{V_{out}(s)}{V_{in}(s)} = H_0 \frac{1+\frac{s}{\omega_z}}{1+\frac{s}{\omega_0 Q}+\left(\frac{s}{\omega_0}\right)^2}$$

$$\omega_z = \frac{1}{r_C C_2} \qquad H_0 = D_0 \frac{R_{load}}{R_{load}+r_L}$$

$$Q = \frac{L_1 C_2 \omega_0 \left(r_C + R_{load}\right)}{L_1 + C_2\left(r_L r_C + r_L R_{load} + r_C R_{load}\right)}$$

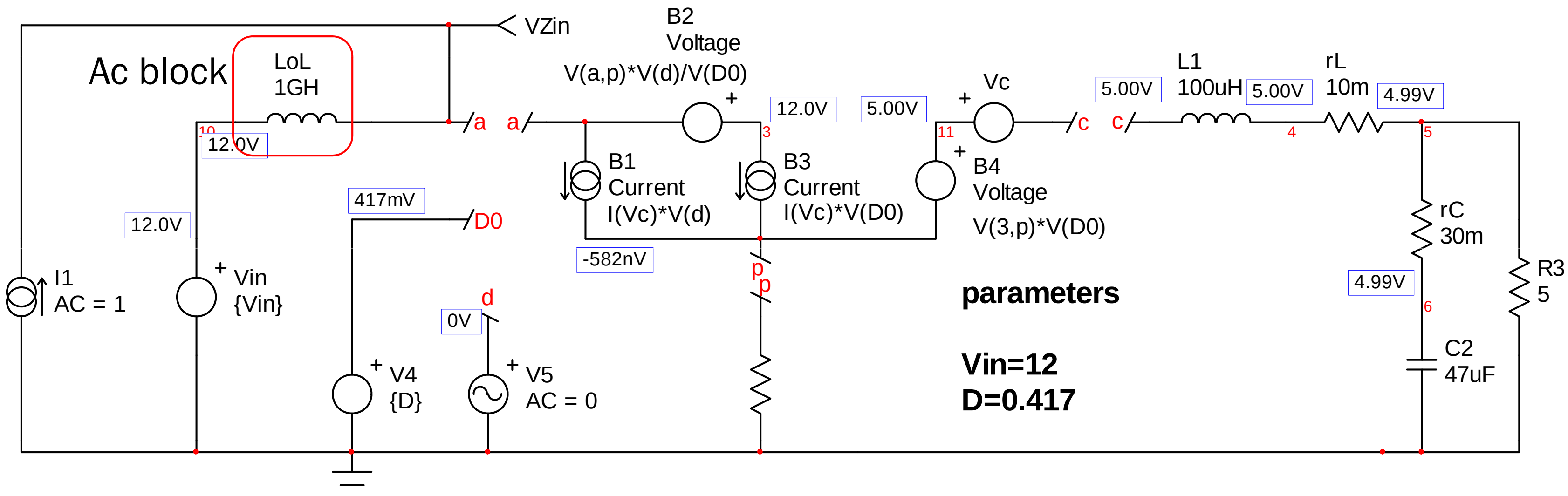
$$\omega_0 = \frac{1}{\sqrt{L_1 C_2}}\sqrt{\frac{r_L + R_{load}}{r_C + R_{load}}}$$



Input-to-output

Buck Input Impedance

- Inductance LoL lets you sweep the input to have Z_{in}



- In this mode, $\hat{d}$ is equal to zero

$$\left.\frac{V_{in}(s)}{I_{in}(s)}\right|_{\hat{d}=0}$$

Source B2 and B1 are zero

Node *p* is ground

$V(a,p) = V_{in}$

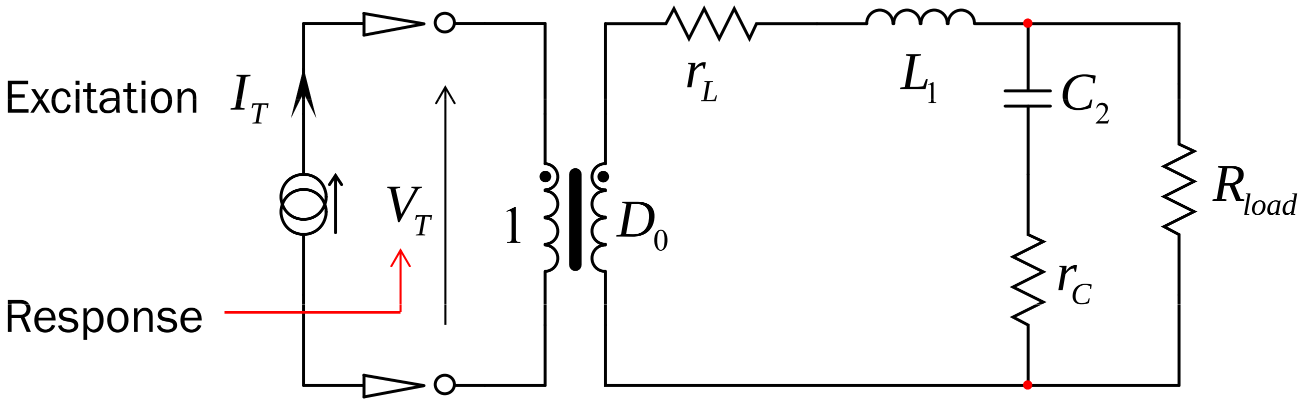
Simplify schematic

Check ac response

Input impedance

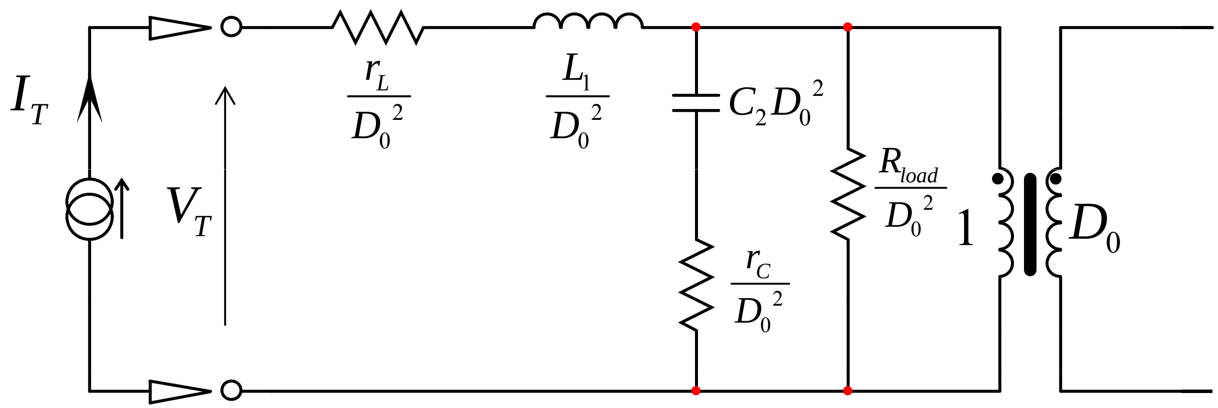
Simplifying and Rearranging is Key

- Install the dc transformer to obtain Z_{in}



$$Z_{in}(s) = \frac{V_T(s)}{I_{in}(s)}$$

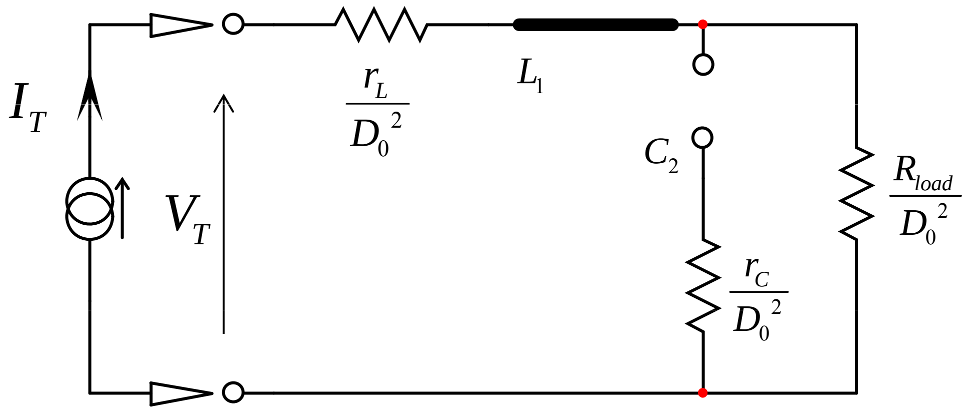
- Reflect elements to the primary side



Input impedance

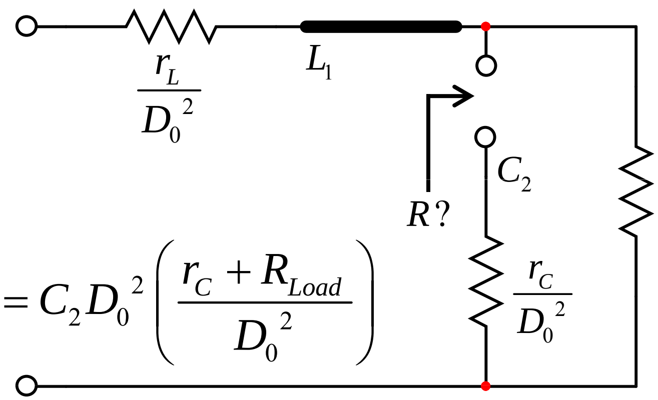
Start with $s = 0$

- Short the inductor, open the capacitor

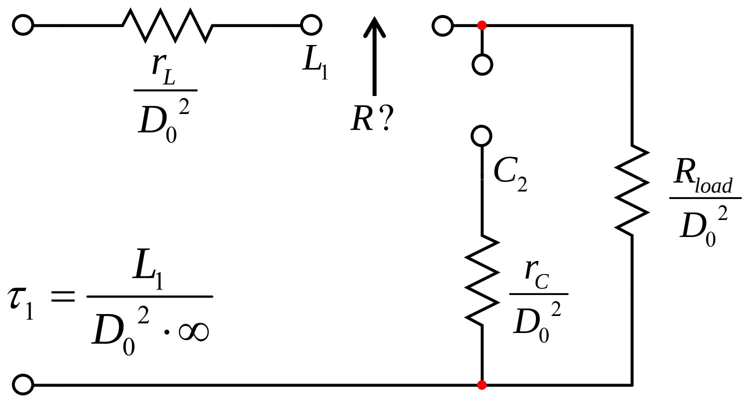


$$R_0 = \frac{r_L + R_{load}}{D_0^2}$$

- For the time constants, suppress the excitation, $I_T = 0$



$$\tau_2 = C_2 D_0^2 \left(\frac{r_C + R_{Load}}{D_0^2} \right)$$

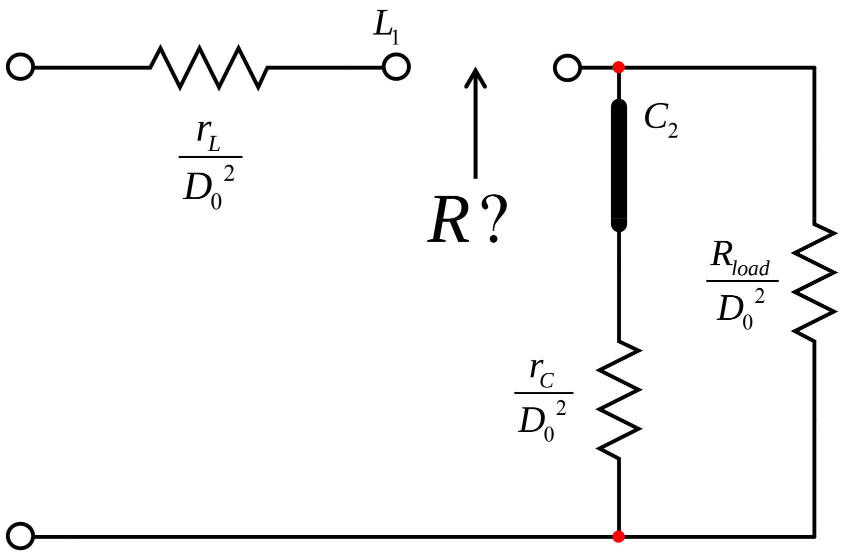


$$\tau_1 = \frac{L_1}{D_0^2 \cdot \infty}$$

Input impedance

Higher Order Coefficients

- Avoid indeterminacy with τ_1 : use τ_2 instead
- Determine τ_1^2



τ_1^2 → High-frequency state; ← $R\,?$

$$\tau_1^2 = \frac{L_1}{D_0{}^2 \cdot \infty}$$

$$\tau_2 \tau_1^2 = C_2 D_0{}^2 \left(\frac{r_C + R_{Load}}{D_0{}^2} \right) \frac{L_1}{D_0{}^2 \cdot \infty} = 0$$

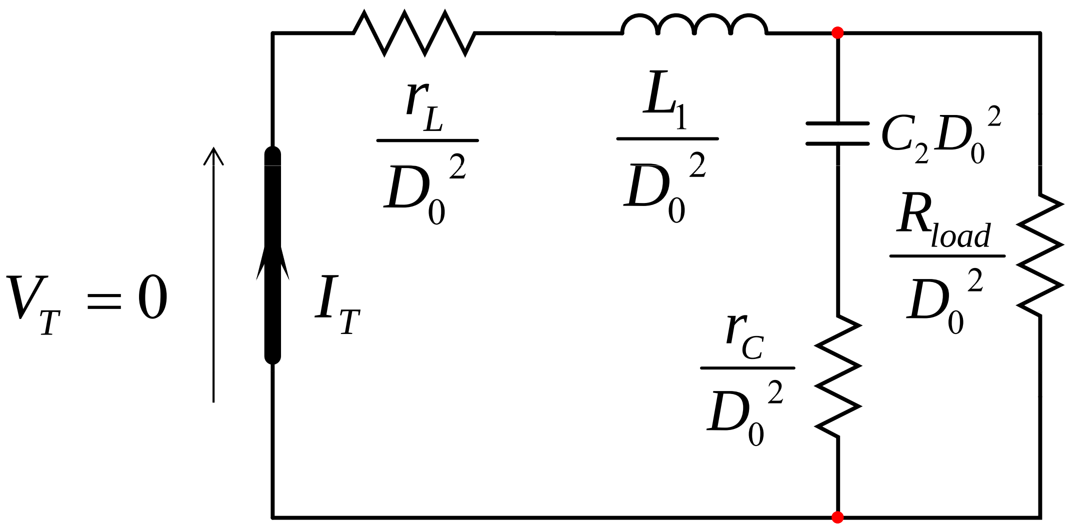
$$\Rightarrow \quad D(s) = 1 + \left[C_2 D_0{}^2 \left(\frac{r_C + R_{Load}}{D_0{}^2} \right) + \frac{L_1}{D_0{}^2 \cdot \infty} \right] s = 1 + sC_2 \left(r_C + R_{Load} \right)$$

Input impedance

The Numerator is Already Known

- Null the response across the current source

  ⟶ Degenerate case, short the generator's terminals!



Same network structure as in slide 114!

$$N(s) = 1 + s\left(\frac{L_1}{r_L + R_{load}} + C_2\left[\left(r_L \,\|\, R_{load}\right) + r_C\right]\right) + s^2\left(L_1 C_2 \frac{r_C + R_{load}}{r_L + R_{load}}\right)$$

Input impedance

Assemble the Pieces

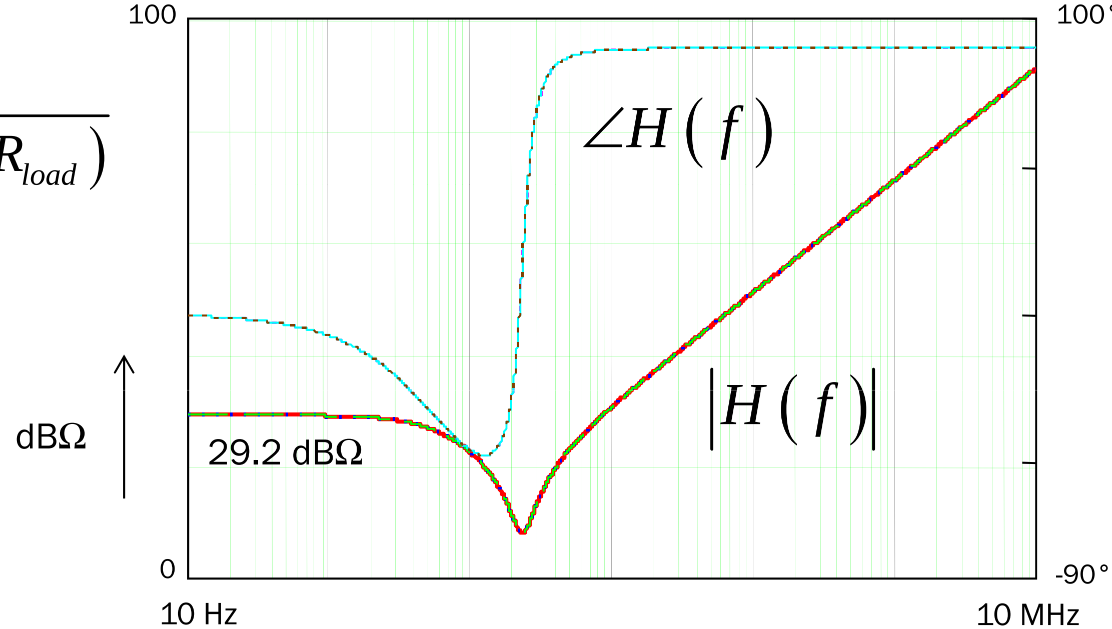
- The transfer function dimension is now in ohms

$$Z_{in}(s) = R_0 \frac{1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2}{1 + \frac{s}{\omega_p}}$$

$$\omega_p = \frac{1}{(r_C + R_{Load})C_2} \qquad R_0 = \frac{r_L + R_{load}}{D_0^{\ 2}}$$

$$Q = \frac{L_1 C_2 \omega_0 (r_C + R_{load})}{L_1 + C_2 (r_L r_C + r_L R_{load} + r_C R_{load})}$$

$$\omega_0 = \frac{1}{\sqrt{L_1 C_2}} \sqrt{\frac{r_L + R_{load}}{r_C + R_{load}}}$$

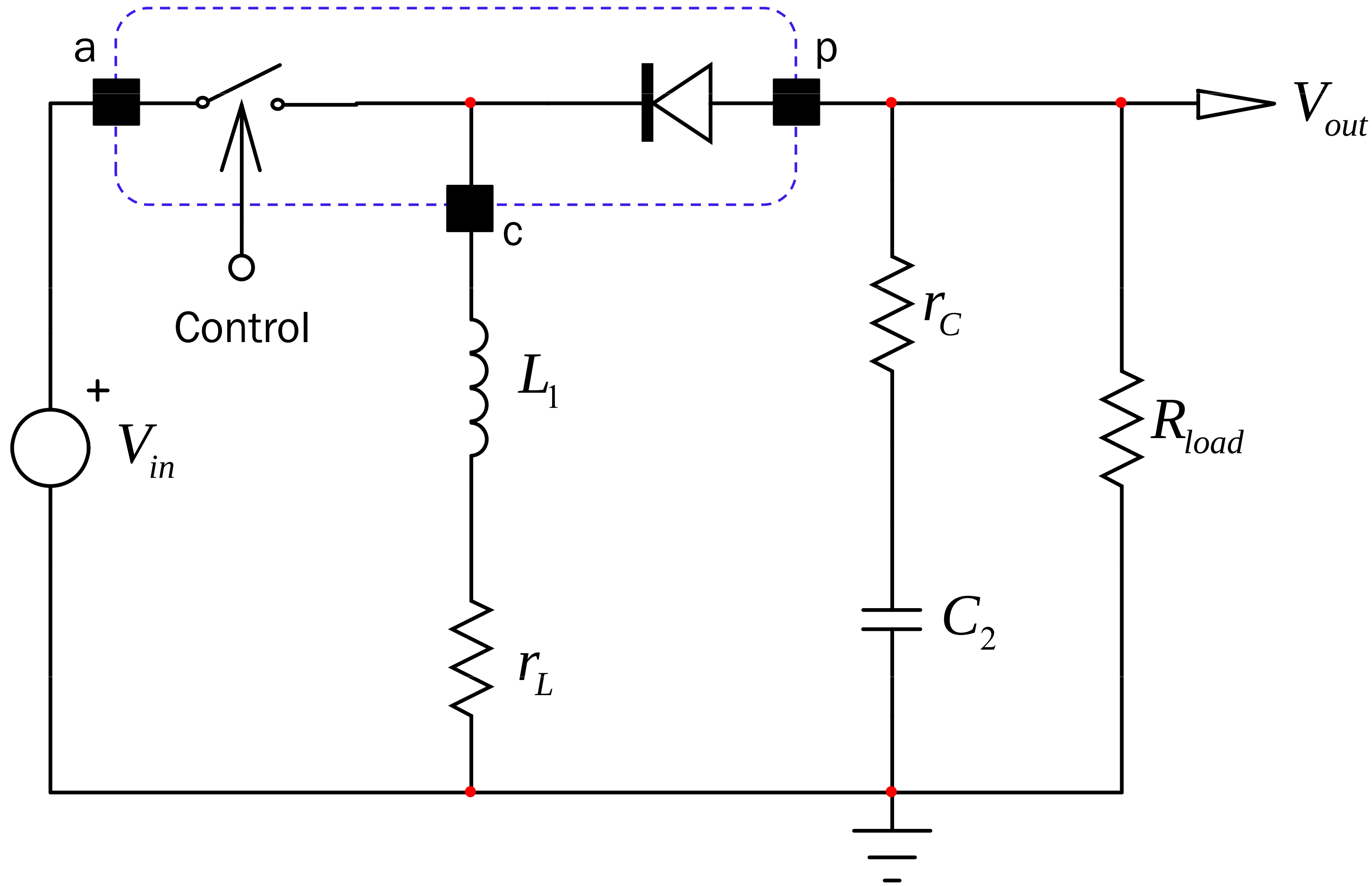


Course Agenda

- ❑ What is a Transfer Function?
- ❑ Why do We Need New Analytical Techniques?
- ❑ Time Constants and Poles
- ❑ Identifying the Zeros
- ❑ The Null Double Injection
- ❑ 2nd-Order Networks
- ❑ The PWM Switch Model
- ❑ A CCM Buck in Voltage Mode
- ❑ **A CCM Buck-Boost in Voltage Mode**

A Buck-Boost Converter

- Replace the switch/diode by the PWM switch model



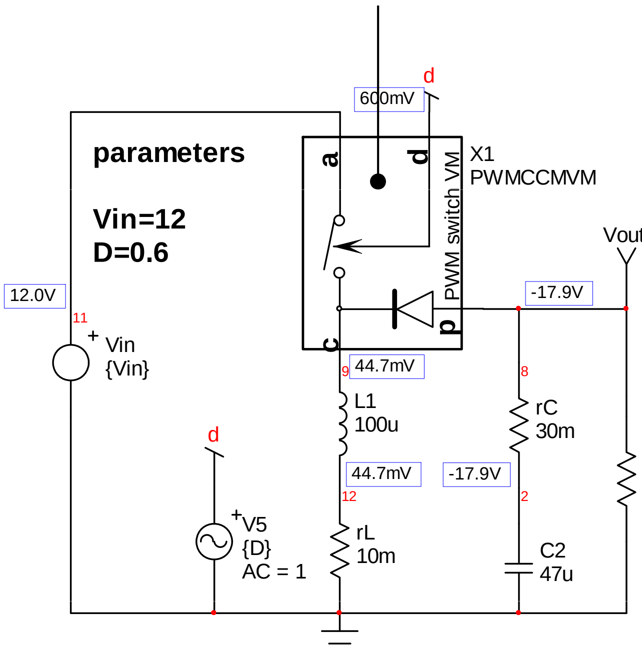
Modeling Switches Only

❑ The PWM switch is invariant in small and large signals

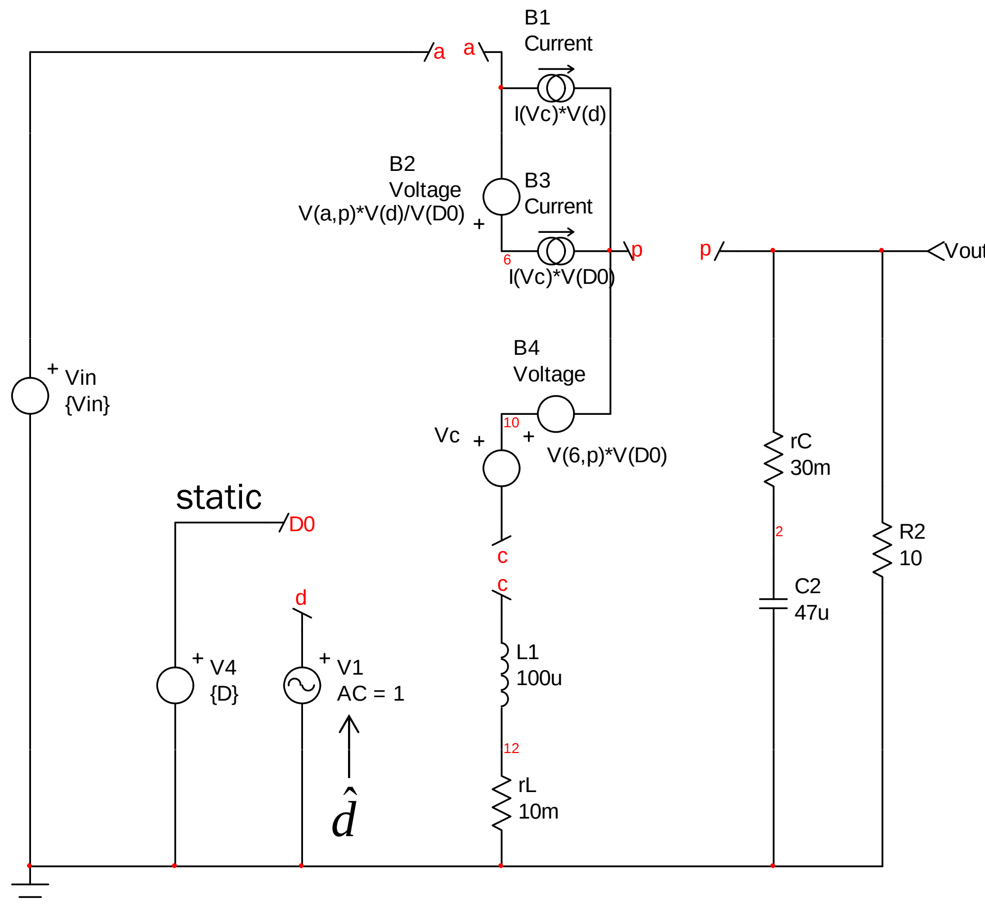
Replace by
small-signal model

parameters

Vin=12
D=0.6



Reference circuit



❑ Always check simplifications versus reference circuit

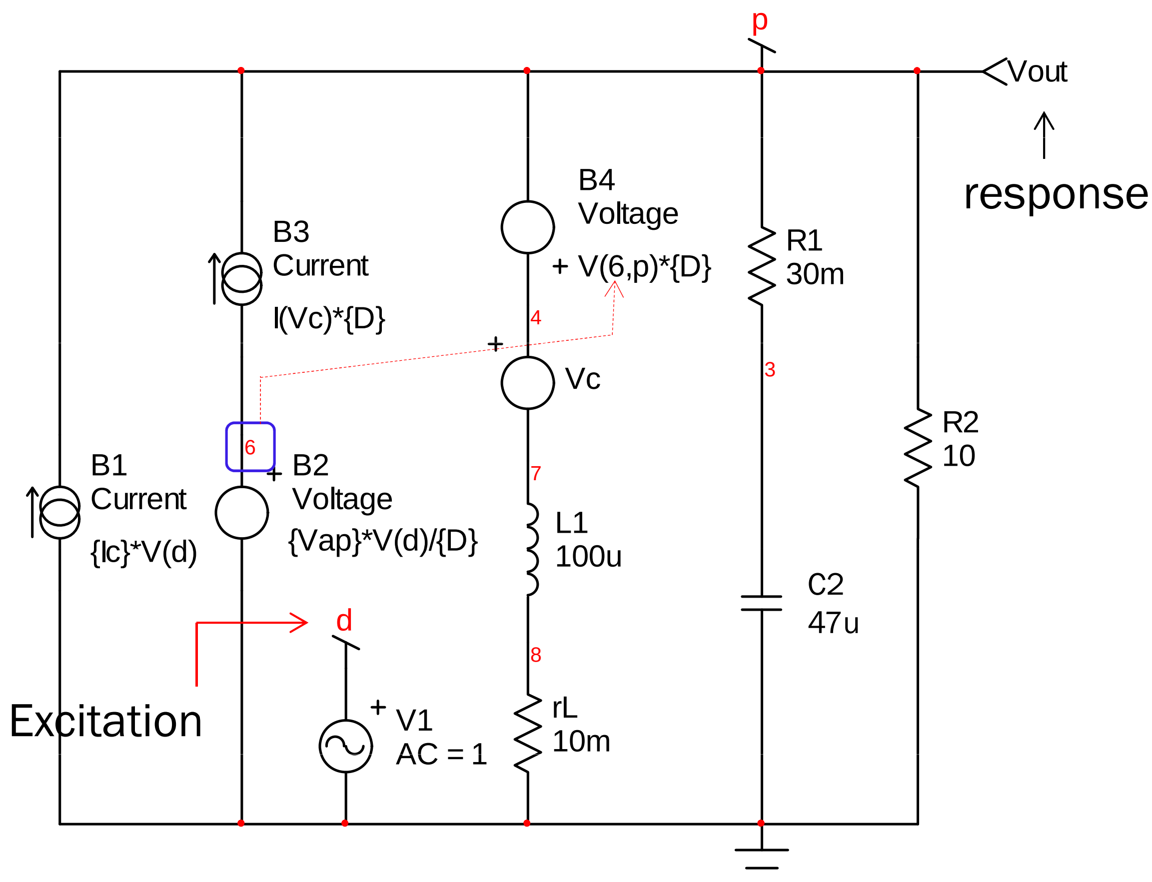
Control to Output Transfer Function

- We want the control-to-output transfer function

parameters

Vin=12
D=0.6
Vout=-Vin*D/(1-D)
RL=10
Vap=Vin-Vout
Ic=-Vout/(RL*(1-D))

$$\left.\frac{V_{out}(s)}{D(s)}\right|_{\hat{v}_{in}=0}$$



- Simplify circuit and check ac response is unchanged

Control-to-output

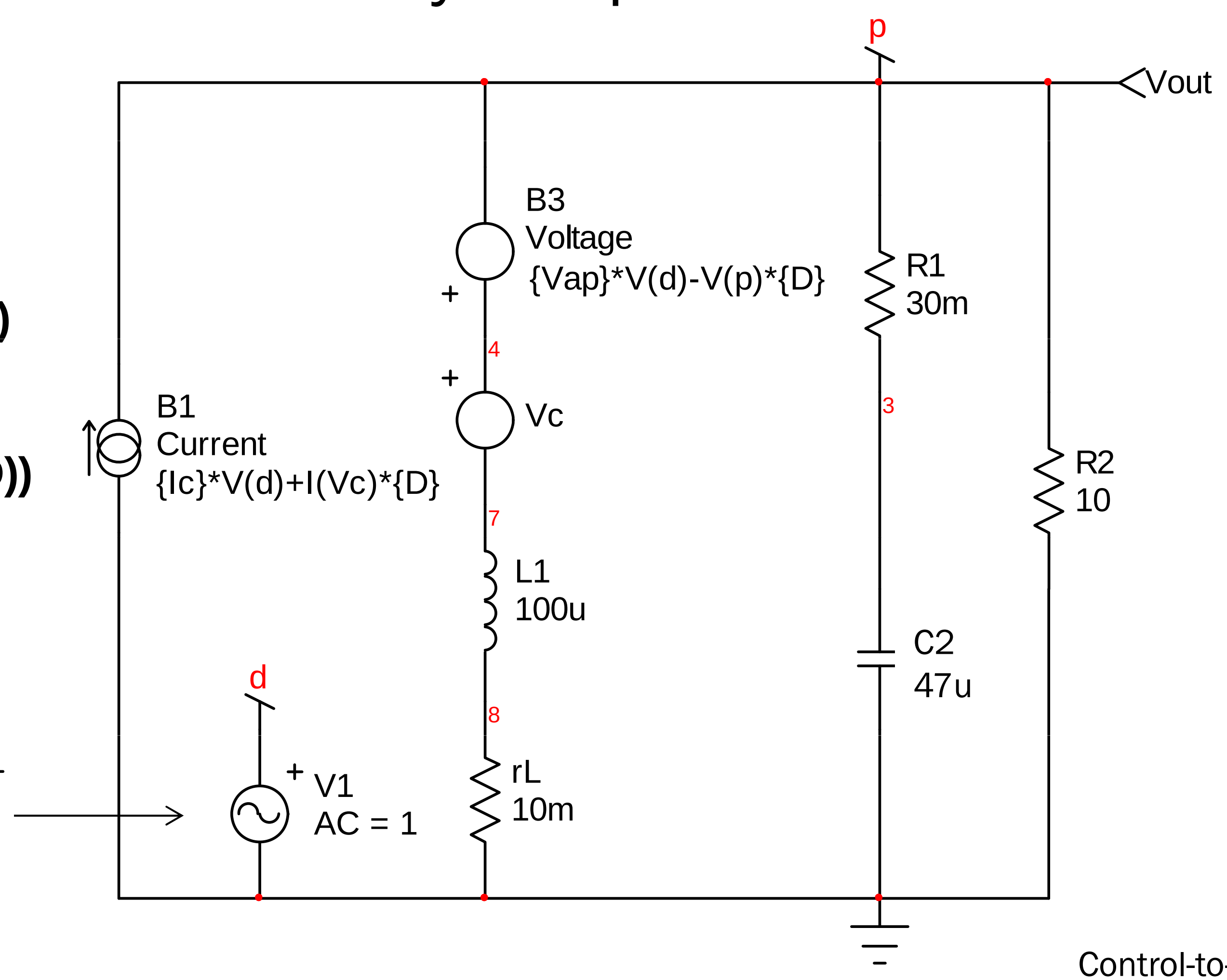
Simplify and Rearrange Expressions

- The final schematic is truly compact

parameters

Vin=12
D=0.6
Vout=-Vin*D/(1-D)
RL=10
Vap=Vin-Vout
Ic=-Vout/(RL*(1-D))

response → $\dfrac{V_{out}(s)}{D(s)}$
excitation →



Control-to-output

A Two-Storage Element Circuit

- There are two independent state variables
  - This is a 2nd–order network

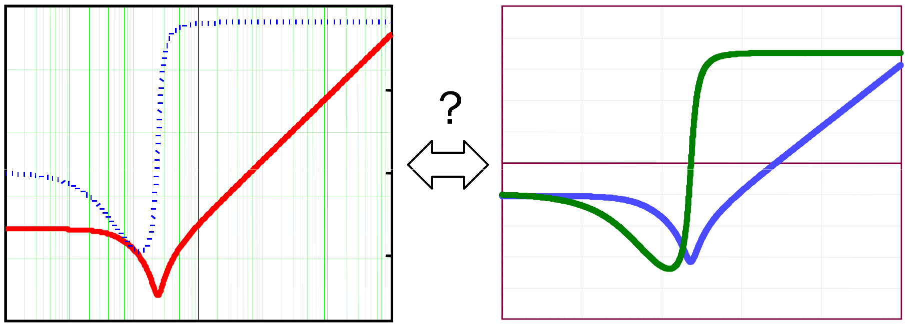
$$H(s) = H_0 \frac{1 + a_1 s + a_2 s^2}{1 + b_1 s + b_2 s^2}$$

1. Determine the dc gain H_0: open capacitor and short inductor
2. b_1 equals the sum of the time constants when excitation is off
3. b_2 combines time constants product when excitation is off

⇨ Assemble $D(s)$

1. Determine the zeros
   - NDI or inspection

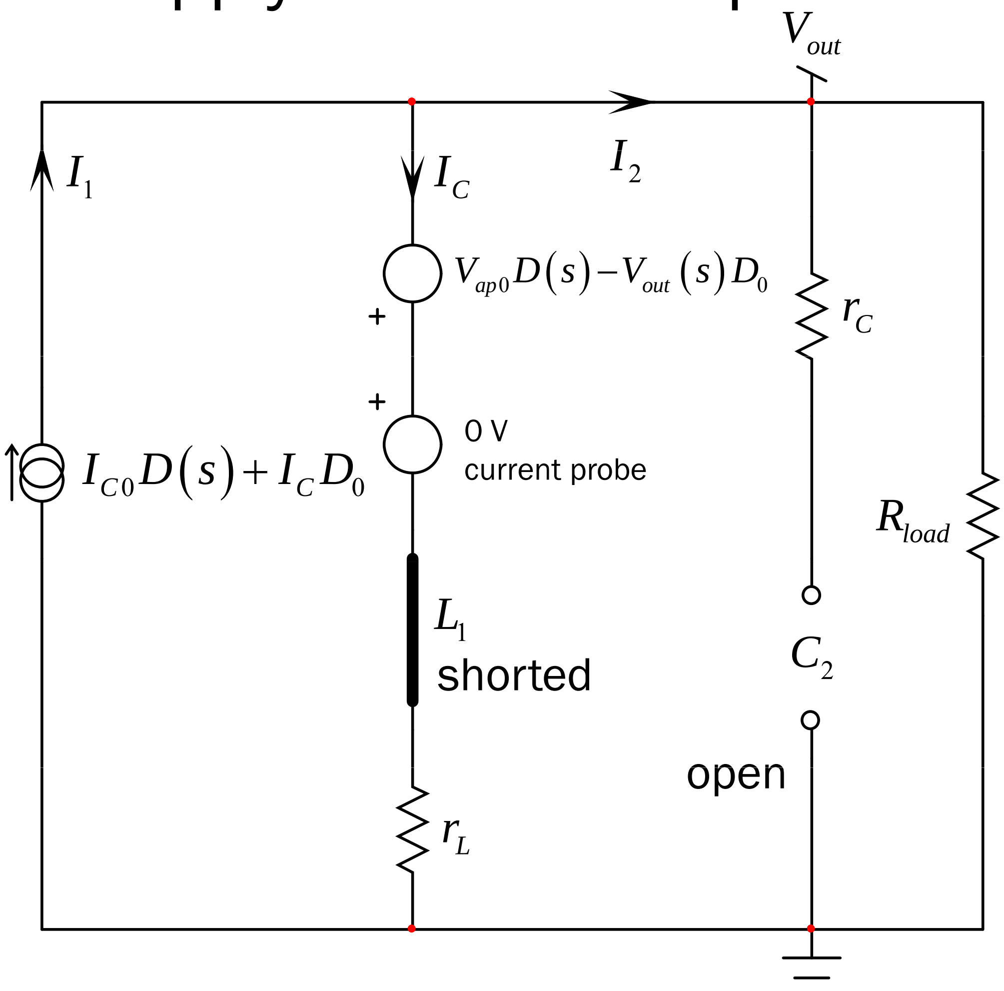
⇨ Assemble $N(s)$



Mathcad® and SPICE agree?

Three Equations for the dc Gain

- Apply KCL on a simple circuit without reactances, $s = 0$



$$I_C(s) = \frac{V_{out}(s) + V_{ap0}D(s) - V_{out}(s)D_0}{r_L}$$

$$I_2(s) = I_{C0}D(s) + I_C(s)D_0 - I_C(s)$$

$$V_{out}(s) = I_2(s)R_{load} \qquad V_{ap0} = V_{in} - V_{out}$$

Substitute
rearrange

$$H_0 = -\left(\frac{1}{1-D_0}\right)\frac{V_{out}r_L + R_{load}(1-D_0)^2(V_{in} - V_{out})}{r_L + R_{load}(1-D_0)^2}$$

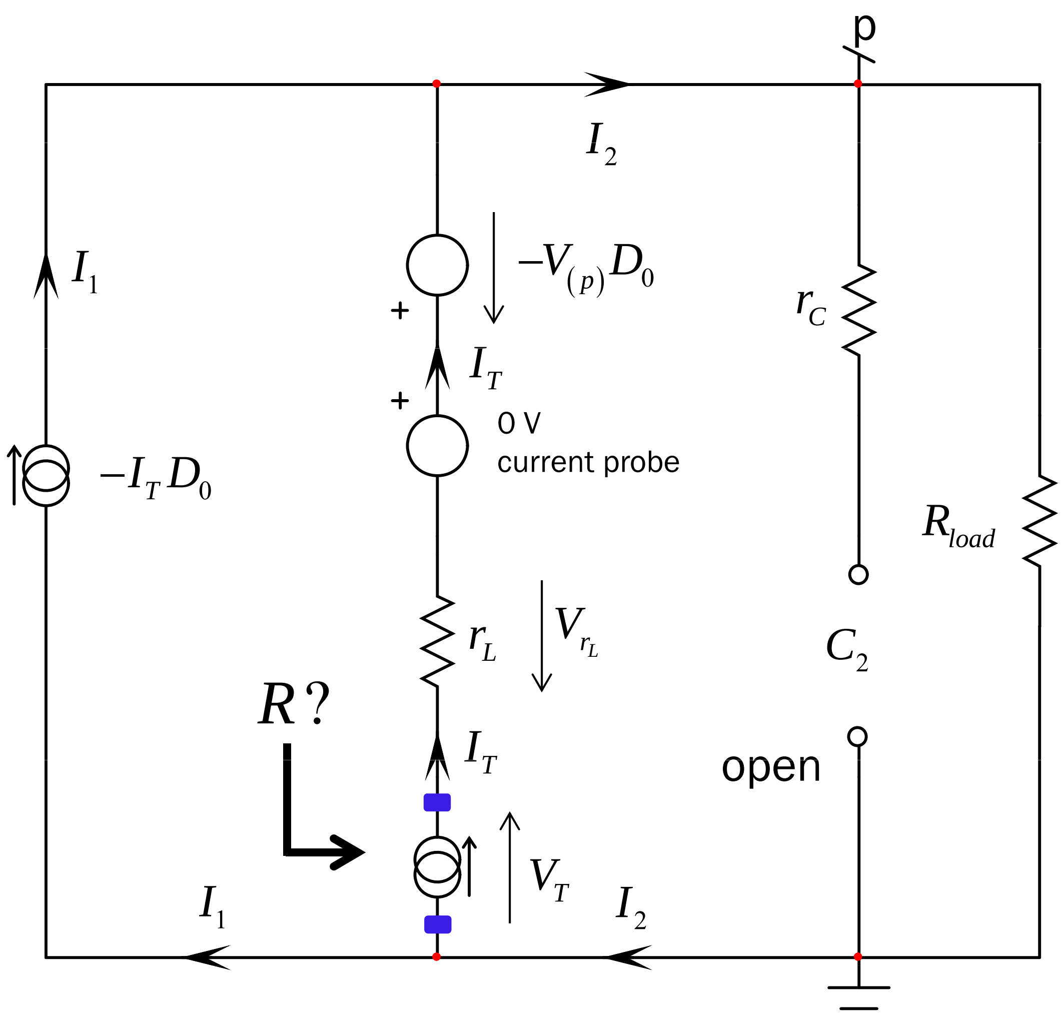
$$r_L \to 0, V_{out} = -V_{in}\frac{D_0}{1-D_0}$$

$$H_0 \approx -\frac{V_{in}}{(1-D_0)^2}$$

Control-to-output

Excitation is Turned off - τ_1

- All expressions featuring $\hat{d}$ or $D(s)$ are set to 0



$$V_{(p)} = I_2 R_{load} = \left(I_1 + I_T\right) R_{load}$$

$$V_{(p)} = I_T \left(1 - D_0\right) R_{load}$$

$$V_T = V_{(p)} - V_{(p)} D_0 + r_L I_T$$

Substitute rearrange

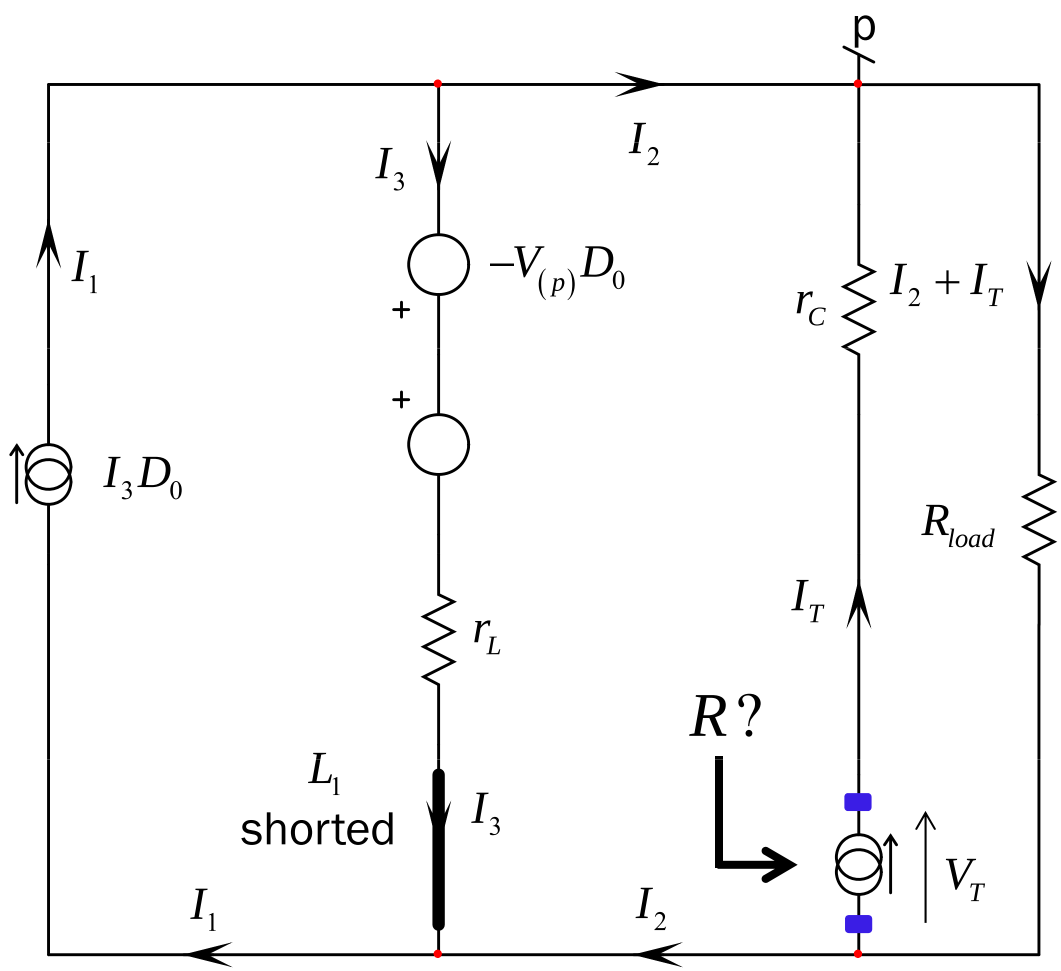
$$R = \frac{V_T}{I_T} = r_L + \left(1 - D_0\right)^2 R_{load}$$

$$\tau_1 = \frac{L_1}{r_L + \left(1 - D_0\right)^2 R_{load}}$$

Control-to-output

Excitation is Turned off - τ_2

- The inductor is now shorted



$$I_2 = I_1 - I_3 = I_3\left(D_0 - 1\right)$$

$$I_3 = \frac{V_{(p)} - V_{(p)}D_0}{r_L} = \frac{V_{(p)}\left(1-D_0\right)}{r_L}$$

$$V_{(p)} = R_{load}\left(I_2 + I_T\right)$$

$$V_T = I_T r_C + V_{(p)}$$

Substitute
rearrange

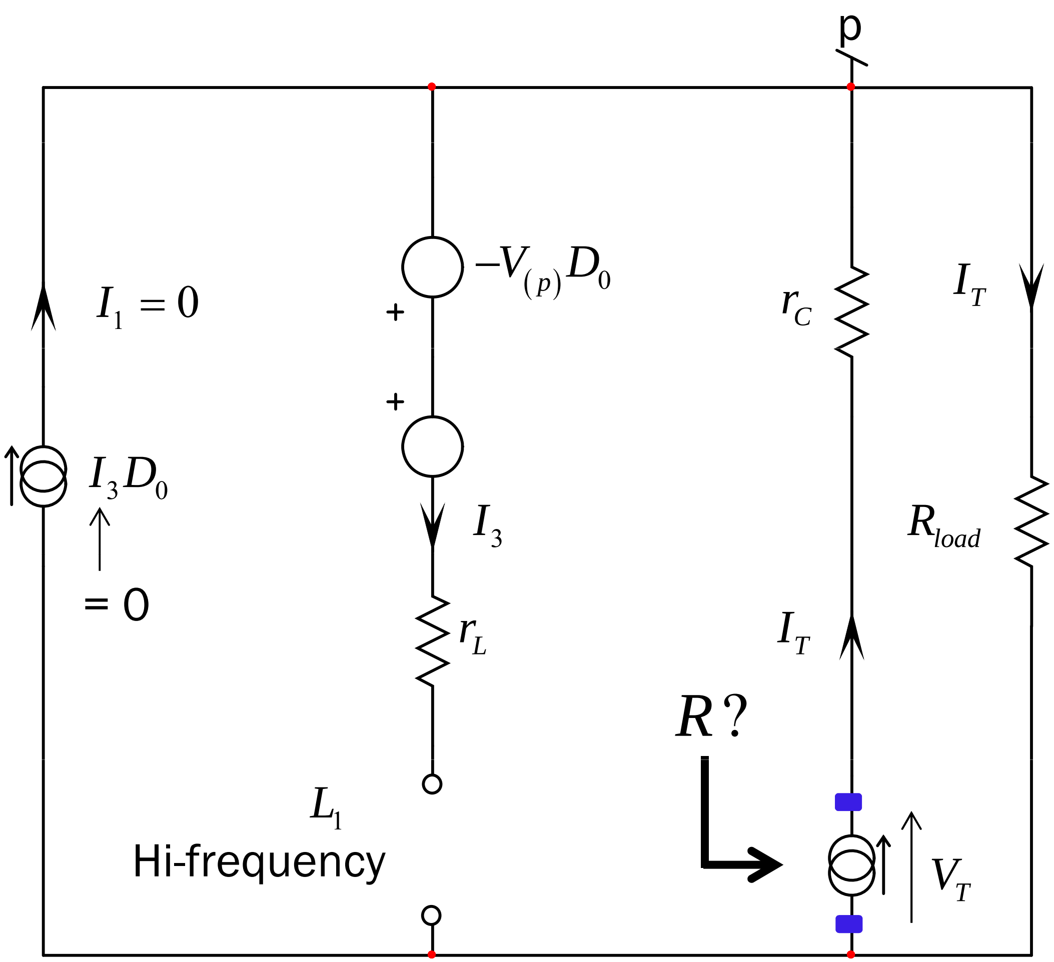
$$R = \frac{V_T}{I_T} = r_C + \frac{R_{load}\, r_L}{R_{load}\left(1-D_0\right)^2 + r_L}$$

$$\tau_2 = C_2\left(r_C + \frac{R_{load}\, r_L}{R_{load}\left(1-D_0\right)^2 + r_L}\right)$$

Control-to-output

Which Combination: $\tau_1\tau_2^1$ or $\tau_2\tau_1^2$?

- Open the inductor: simplest configuration is τ_2^1



$$\tau_2^1 = C_2\left(r_C + R_{load}\right)$$

Combine with τ_1

$$\tau_1 = \frac{L_1}{r_L + \left(1 - D_0\right)^2 R_{load}}$$

$$b_2 = \tau_1\tau_2^1 = \frac{L_1}{r_L + \left(1 - D_0\right)^2 R_{load}} C_2\left(r_C + R_{load}\right)$$

Control-to-output

Denominator Expression

- The 2nd-order denominator can be formed

$$b_1 = \tau_1 + \tau_2 = \frac{L_1}{r_L + (1-D_0)^2 R_{load}} + C_2\left(r_C + \frac{R_{load} r_L}{R_{load}(1-D_0)^2 + r_L}\right)$$

$$b_2 = \tau_1 \tau_2^1 = \frac{L_1}{r_L + (1-D_0)^2 R_{load}} C_2 (r_C + R_{load})$$

$$D(s) = 1 + \left[\frac{L_1}{r_L + (1-D_0)^2 R_{load}} + C_2\left(r_C + \frac{R_{load} r_L}{R_{load}(1-D_0)^2 + r_L}\right)\right] s + \left[\frac{L_1}{r_L + (1-D_0)^2 R_{load}} C_2 (r_C + R_{load})\right] s^2$$

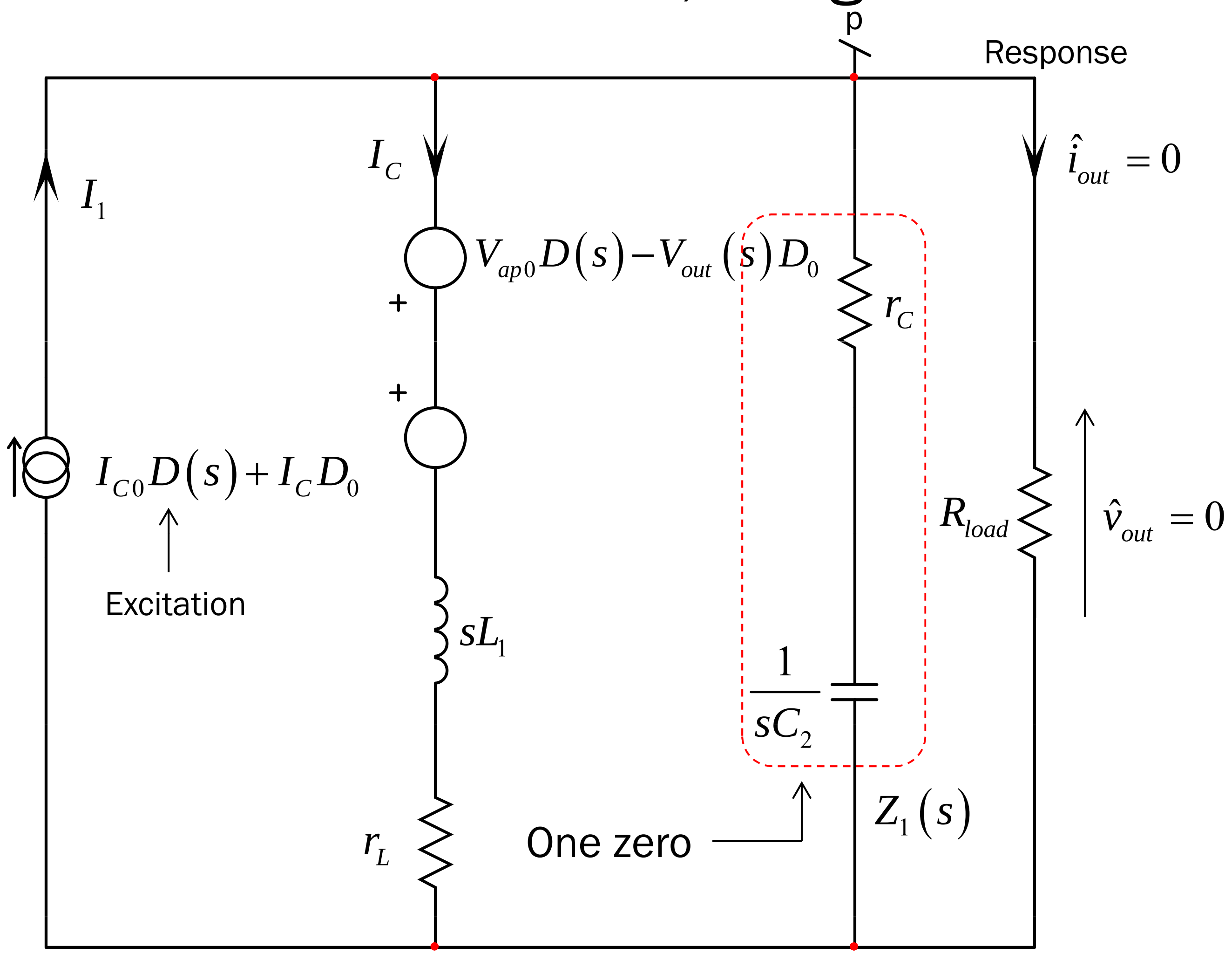
$$\Rightarrow \quad H(s) = H_0 \frac{N(s)}{1 + b_1 s + b_2 s^2}$$

Zeros are missing!

Control-to-output

Determining the Numerator

- To determine zeros, bring the excitation back



"What conditions in the transformed circuit null the response?"

$\hat{i}_{out} = 0$

$$Z_1(s) = r_C + \frac{1}{sC_2} = 0$$

$$I_1 = I_C$$

What if L_1 and C_2 are in HF state?

First Zero is Easy

- The equivalent series resistance brings the first zero

$$Z_1(s_z) = r_C + \frac{1}{sC_2} = 0 \longrightarrow Z_1(s_z) = \frac{1 + sr_C C_2}{sC_2} = 0$$

- The negative (LHP) root is simply

$$s_{z_1} = -\frac{1}{r_C C_2} \longrightarrow \omega_{z_1} = \frac{1}{r_C C_2}$$

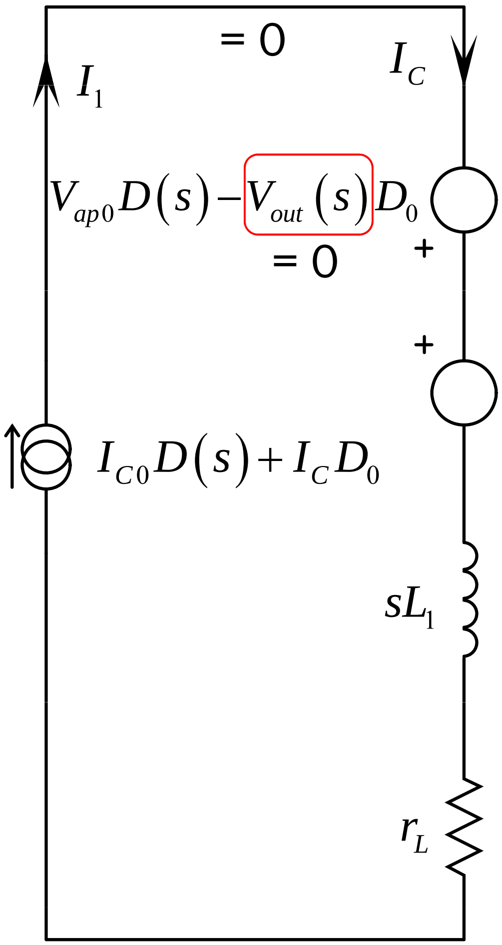
- Almost there...

$$H(s) = H_0 \frac{\left(1 + \frac{s}{\omega_{z_1}}\right)(\ldots)}{1 + b_1 s + b_2 s^2}$$

Control-to-output

Equate Current Expressions

- The output null implies that $\hat{v}_{out} = 0$



$$I_1 - I_C = 0 \qquad I_C(s) = \frac{V_{ap0}D(s)}{sL_1 + r_L}$$

$$I_1(s) = I_{C0}D(s) + I_C(s)D_0$$

Substitute I_C in I_1

$$I_{C0}D(s) + \frac{V_{ap0}D(s)}{r_L + sL_1}D_0 - \frac{V_{ap0}D(s)}{r_L + sL_1} = 0$$

Solve for the root

$$s_{z_2} = \frac{(1-D_0)^2 R_{load} - r_L D_0}{D_0 L_1} \xrightarrow[r_L \ll R_{load}]{} \omega_{z_2} \approx \frac{(1-D_0)^2 R_{load}}{D_0 L_1}$$

Positive root, RHPZ!

Control-to-output

Final Expression

- Assemble the pieces to form the transfer function

$$H_0 = -\left(\frac{1}{1-D_0}\right)\frac{V_{out} r_L + R_{load}(1-D_0)^2 (V_{in} - V_{out})}{r_L + R_{load}(1-D_0)^2} \qquad N(s) = (1 + s r_C C_2)\left(1 - s\frac{D_0 L_1}{(1-D_0)^2 R_{load} - r_L D_0}\right)$$

$$D(s) = 1 + \left[\frac{L_1}{r_L + (1-D_0)^2 R_{load}} + C_2\left(r_C + \frac{R_{load} r_L}{R_{load}(1-D_0)^2 + r_L}\right)\right]s + \left[\frac{L_1}{r_L + (1-D_0)^2 R_{load}} C_2 (r_C + R_{load})\right]s^2$$

- Rearrange under a 2nd-order polynomial form

$$\omega_0 = \frac{1}{\sqrt{b_2}} = \frac{1}{\sqrt{\dfrac{L_1}{r_L + (1-D_0)^2 R_{load}} C_2 (r_C + R_L)}} = \frac{1}{\sqrt{L_1 C_2}\sqrt{\dfrac{r_C + R_L}{r_L + (1-D_0)^2 R_{load}}}} \approx \frac{1-D_0}{\sqrt{L_1 C_2}}$$

$$Q = \frac{\sqrt{b_2}}{b_1} = \frac{\sqrt{\dfrac{L_1}{r_L + (1-D_0)^2 R_{load}} C_2 (r_C + R_L)}}{\dfrac{L_1}{r_L + (1-D_0)^2 R_{load}} + C_2\left(r_C + \dfrac{R_{load} r_L}{R_{load}(1-D_0)^2 + r_L}\right)} \approx (1-D_0) R_{load}\sqrt{\frac{C_2}{L_1}}$$

Control-to-output

Plot the Dynamic Response

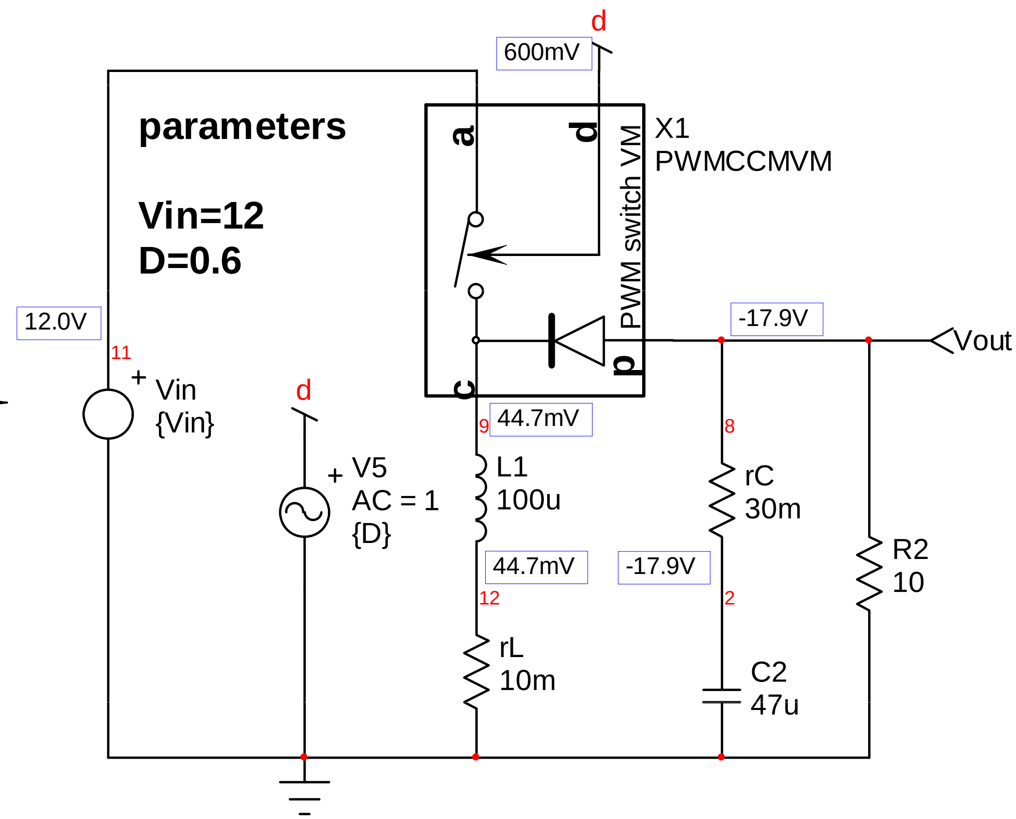
- Check response versus that of PWM switch model

$$H(s) = H_0 \frac{\left(1+\frac{s}{\omega_{z_1}}\right)\left(1-\frac{s}{\omega_{z_2}}\right)}{1+\frac{s}{\omega_0 Q}+\left(\frac{s}{\omega_0}\right)^2}$$

$$H_0 = \frac{V_{in}}{(1-D)^2} \qquad \omega_{z_1} = \frac{1}{r_C C_2}$$

$$\omega_{z_2} = \frac{(1-D)^2 R_{load}}{DL_1}$$

$$\omega_0 = \frac{1-D}{\sqrt{L_1 C_2}} \qquad Q = (1-D) R_{load} \sqrt{\frac{C_2}{L_1}}$$

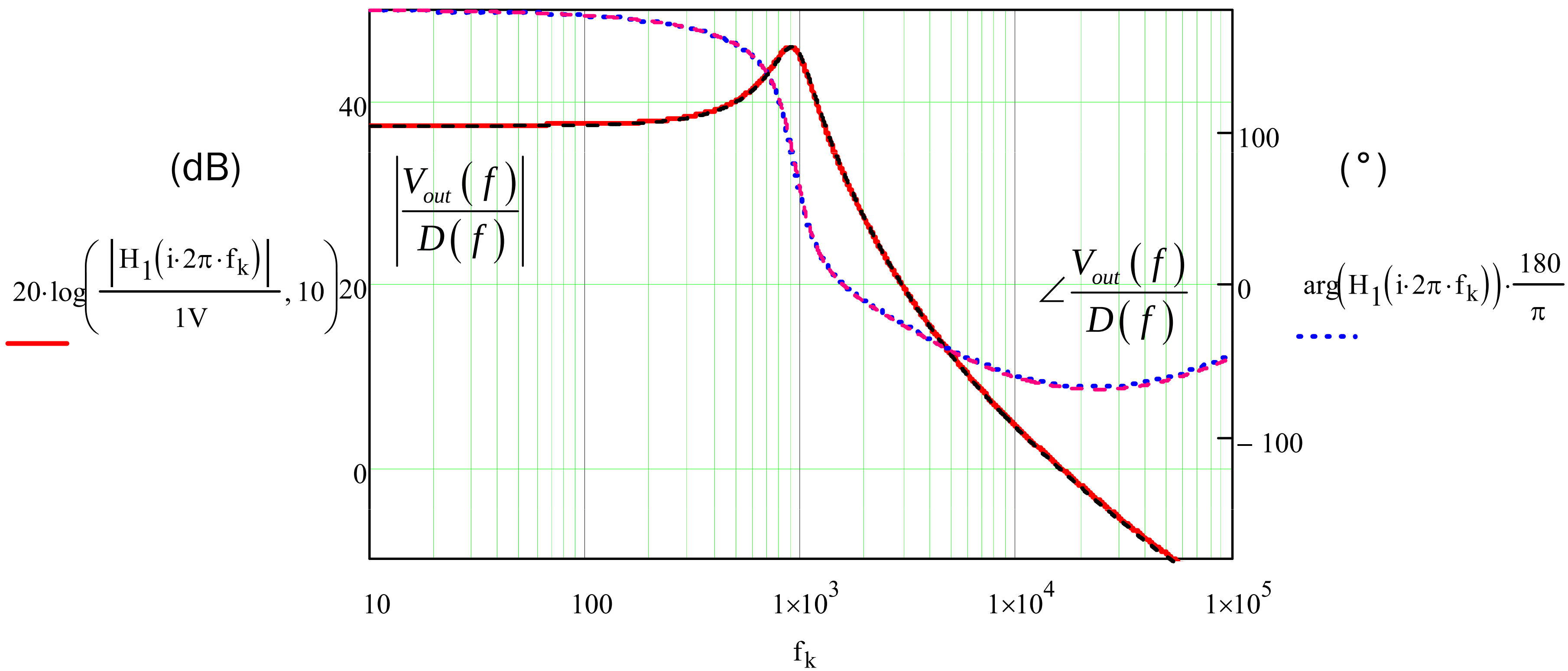


Large-signal PWM switch model

Control-to-output

SPICE and Mathcad® Plots

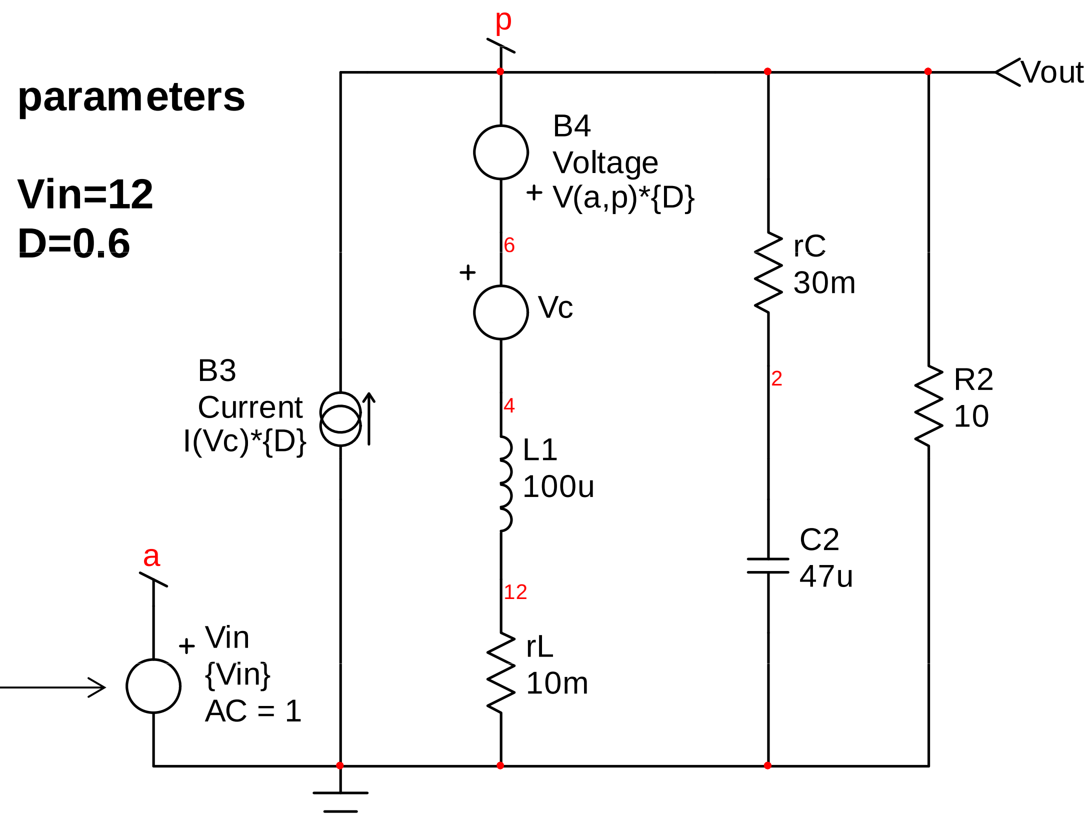
- Curves superimpose: transfer function is correct!



Control-to-output

Input to Output Transfer Function

- This time $\hat{d}$ is 0 and V_{in} is now ac-modulated
- All sources including $\hat{d}$ (d) are set to 0



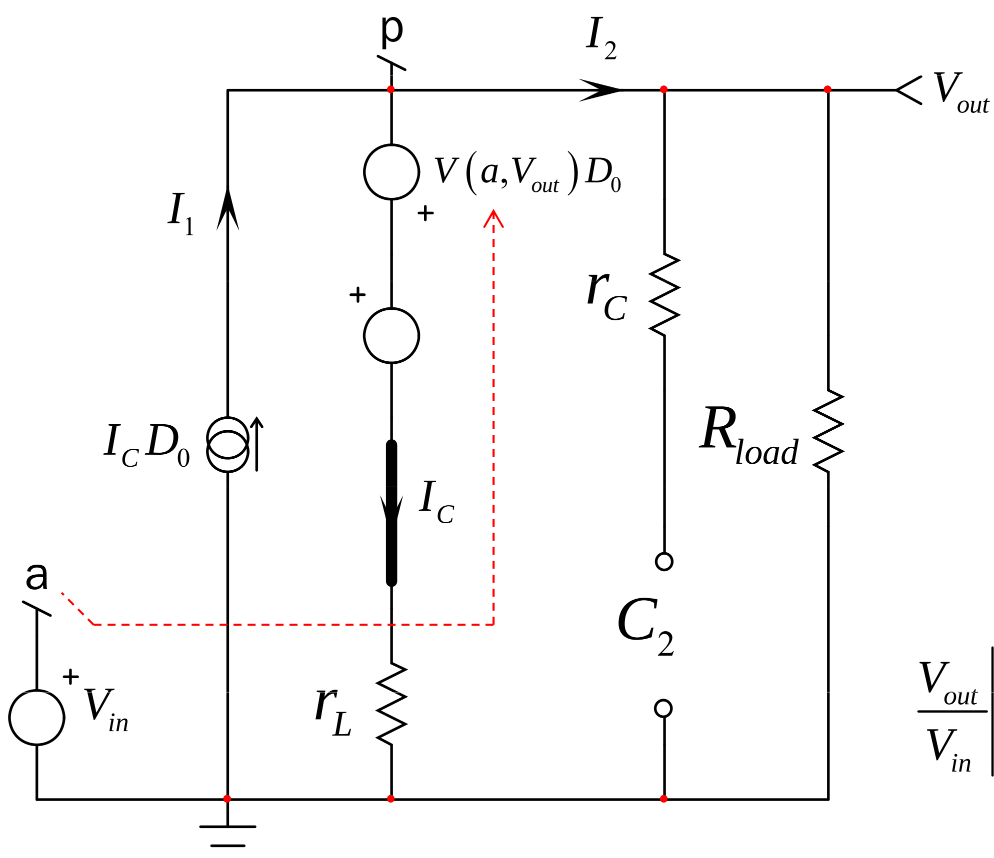
response $\longrightarrow$ $\dfrac{V_{out}(s)}{V_{in}(s)}$
excitation $\longrightarrow$

- Excitation is 0, structure is unchanged: Reuse $D(s)$!

slide 131

Static Gain - Response for s = 0

- Open the capacitor and short the inductor



$$I_C = \frac{V_{out} + V_{in}D_0 - V_{out}D_0}{r_L}$$

$$I_1 = \left(\frac{V_{out} + V_{in}D_0 - V_{out}D_0}{r_L}\right)D_0$$

$$V_{out} = (I_1 - I_C)R_{load}$$

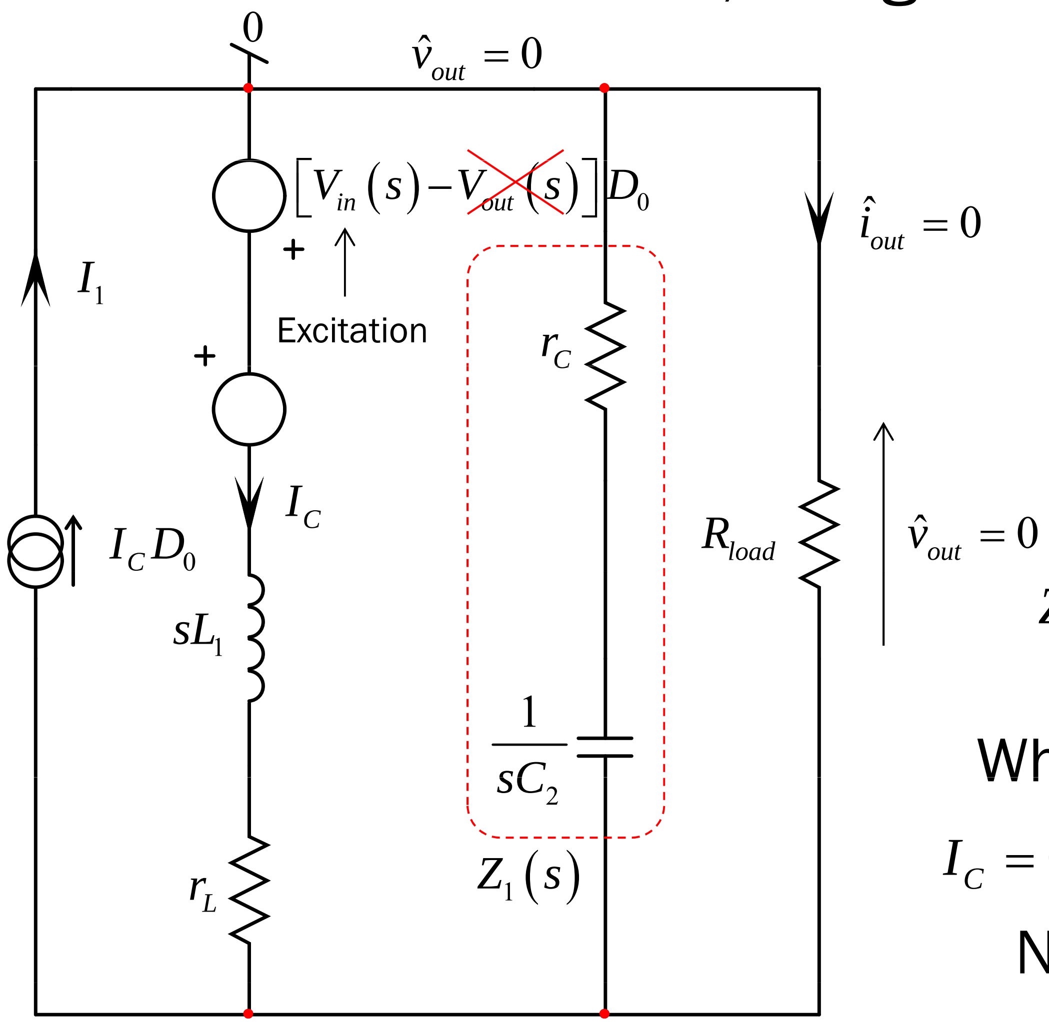
Solve for V_{out} and rearrange

$$\left.\frac{V_{out}}{V_{in}}\right|_{s=0} = H_0 = -\frac{D(1-D)R_{load}}{(1-D)^2 R_{load} + r_L} \approx -\frac{D}{1-D}$$

Input-to-output

Determining the Numerator: Null V_{out}

- To determine zeros, bring the excitation back



"What conditions in the transformed circuit null the response?"

$\hat{i}_{out} = 0$

$$Z_1(s) = r_C + \frac{1}{sC_2} = 0 \longrightarrow \omega_{z_1} = \frac{1}{r_C C_2}$$

What if L_1 is set to its HF state?

$$I_C = 0 \longrightarrow I_C D_0 = 0$$

No response, 1 zero only

Input-to-output

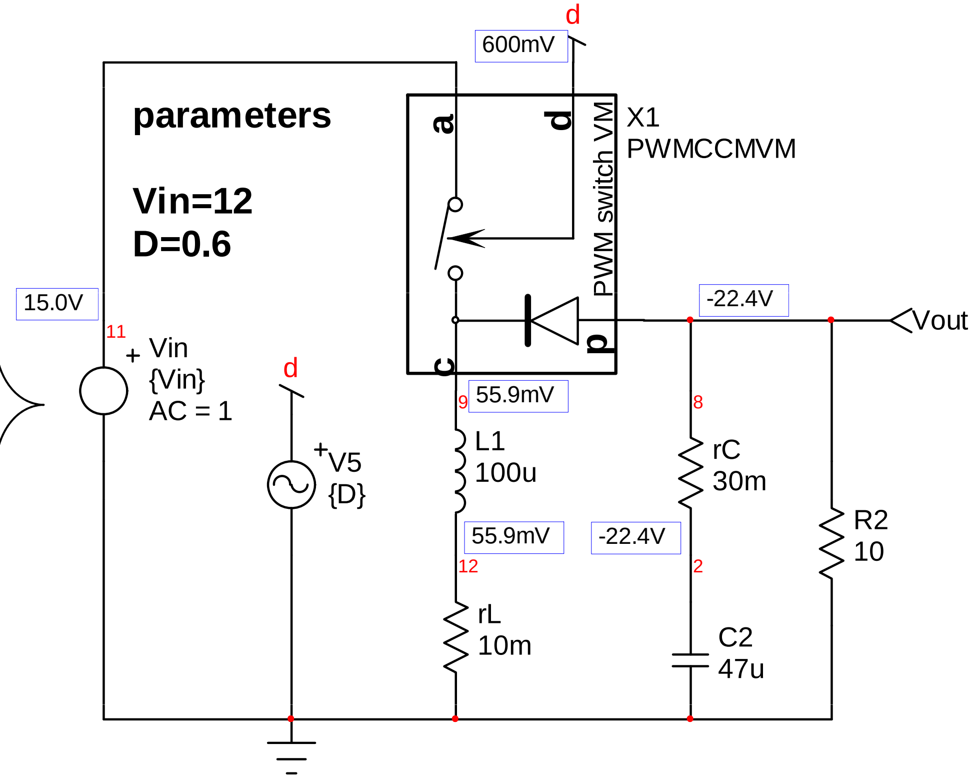
Final Transfer Function

- The transfer function is immediate

$$\frac{V_{out}(s)}{V_{in}(s)} \approx -\left(\frac{D}{1-D}\right)\frac{(1+sr_C C_2)}{1+\dfrac{s}{\omega_0 Q}+\left(\dfrac{s}{\omega_0}\right)^2}$$

$$\omega_0 \approx \frac{1-D}{\sqrt{L_1 C_2}}$$

$$Q \approx (1-D) R_{load}\sqrt{\frac{C_2}{L_1}}$$

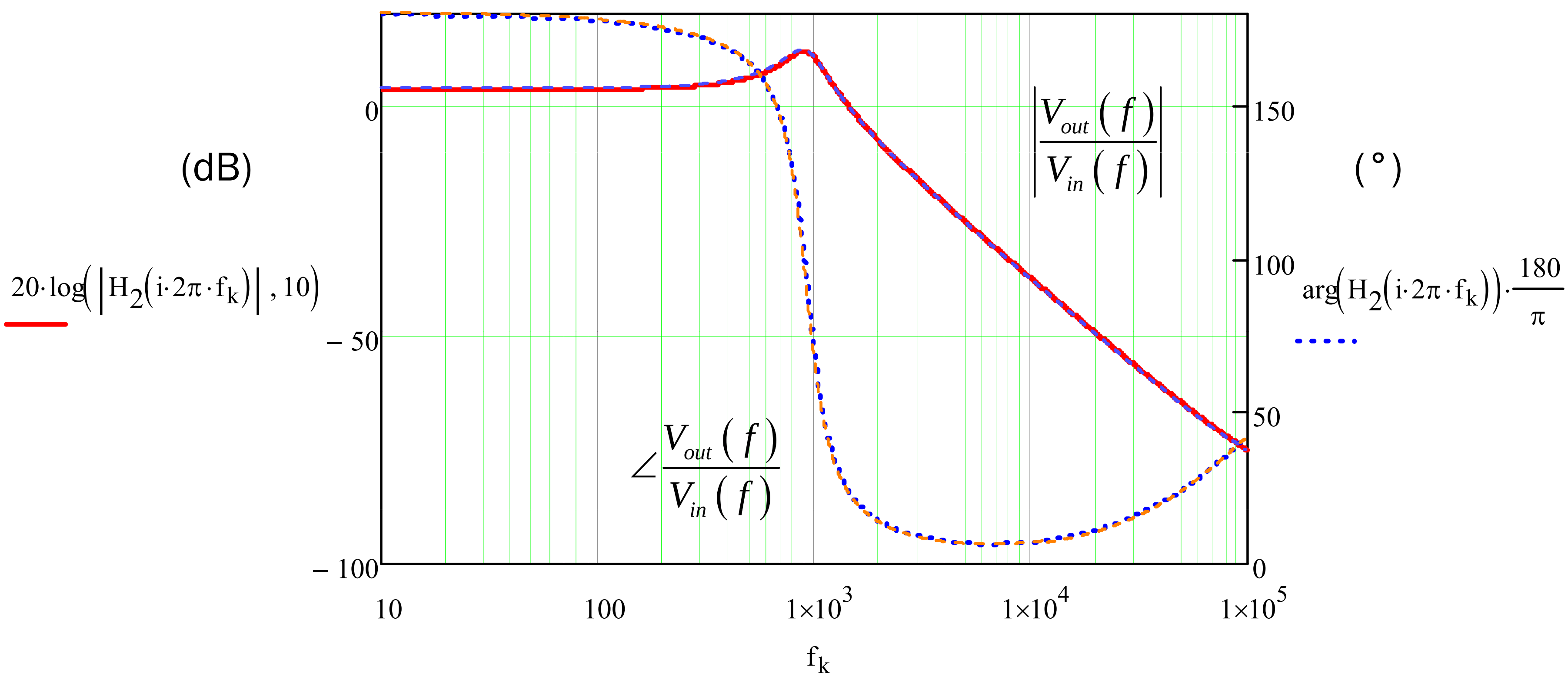


Large-signal PWM switch model

Input-to-output

SPICE and Mathcad® Plots

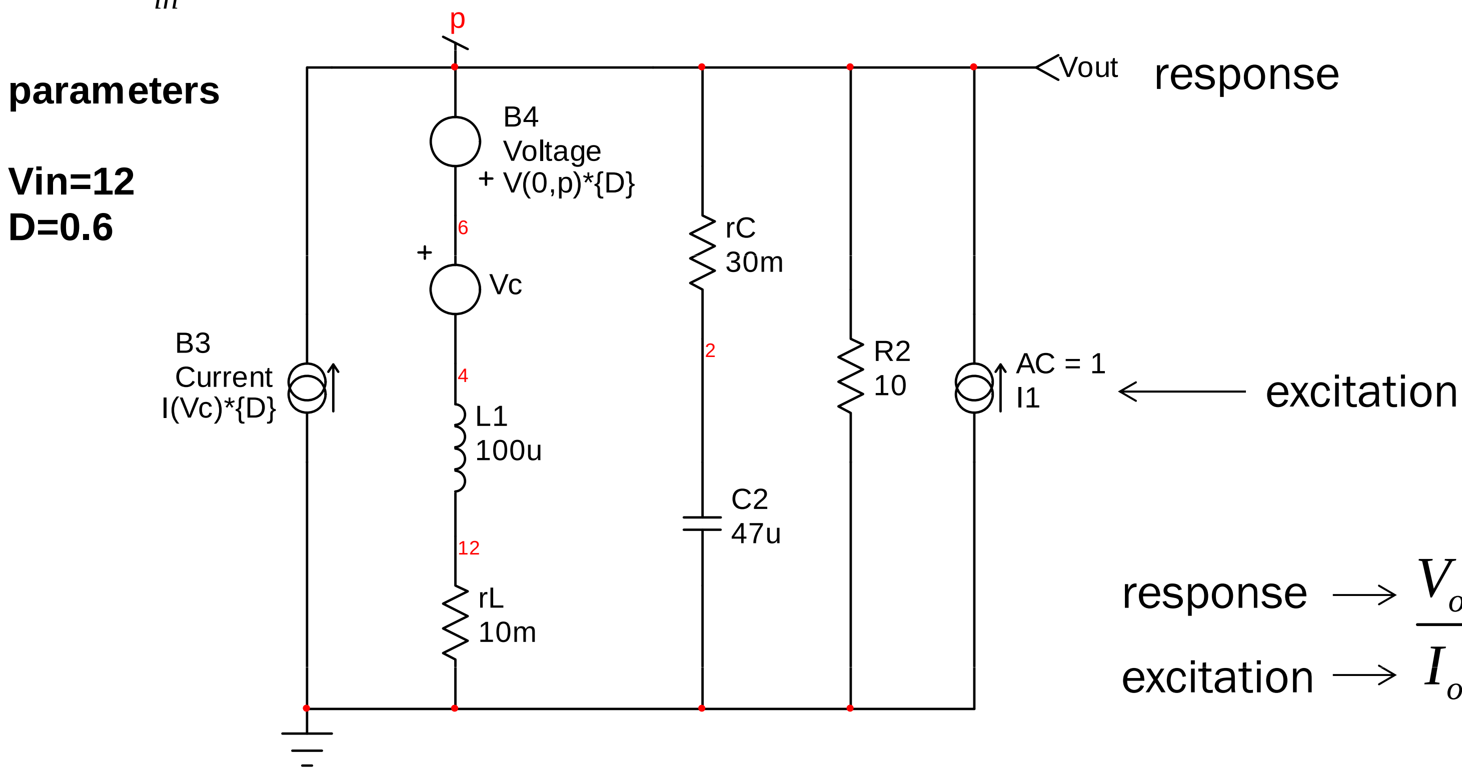
- Curves superimpose: transfer function is correct!



Input-to-output

Output Impedance Determination

- Install an 1-A ac current source on the output
  - $\hat{d} = \hat{v}_{in} = 0$



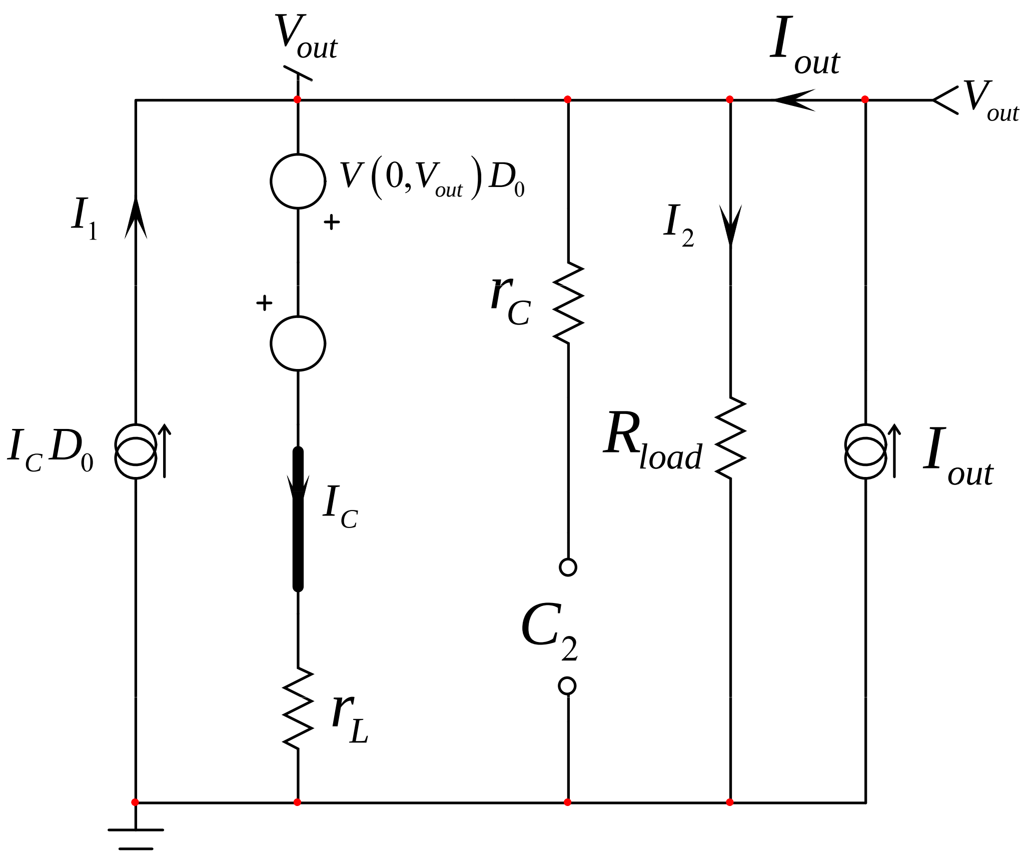
response → $\frac{V_{out}(s)}{I_{out}(s)}$ ← excitation

- If excitation is 0, structure is unchanged. Same $D(s)$!

→ slide 131

Static Resistance: Response for $s = 0$

- ❑ Open the capacitor and short the inductor



$$I_C = \frac{V_{out} - V_{out}D_0}{r_L} = \frac{V_{out}(1-D_0)}{r_L}$$

$$I_1 = \left(\frac{V_{out}(1-D_0)}{r_L}\right)D_0 \qquad I_2 = \frac{V_{out}}{R_{load}}$$

$$I_{out} = I_C + I_2 - I_1$$

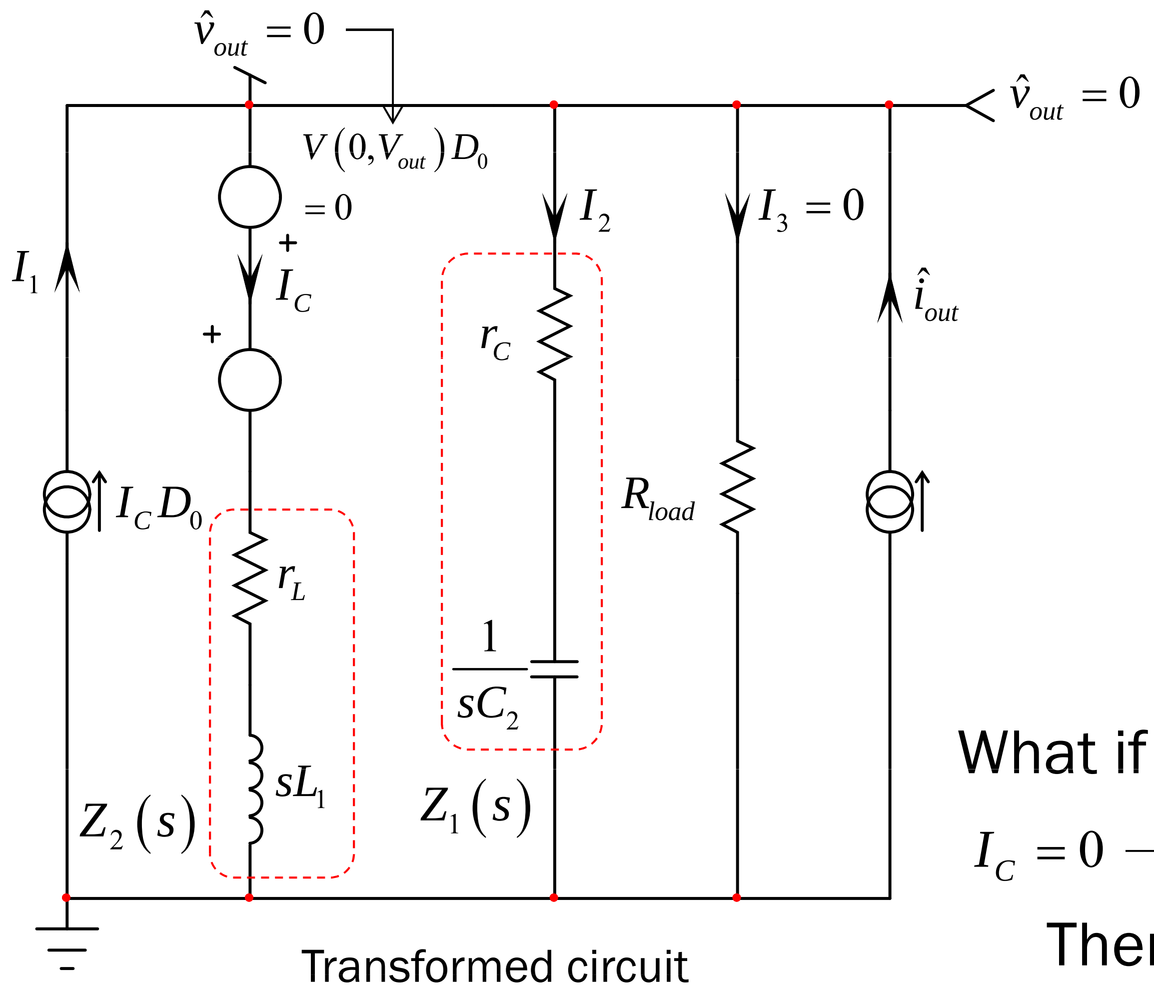
Factor V_{out} and rearrange

$$R_0 = \frac{V_{out}}{I_{out}} = \frac{r_L}{(1-D_0)^2 + \frac{r_L}{R_{load}}}$$

Output impedance

Determining the Numerator

- To determine zeros, bring the excitation back



Transformed circuit

“What conditions in the transformed circuit null the response?”

$$\hat{v}_{out} = 0$$

$$Z_1(s) = r_C + \frac{1}{sC_2} = 0$$

What if L_1 and C_2 are in HF state?

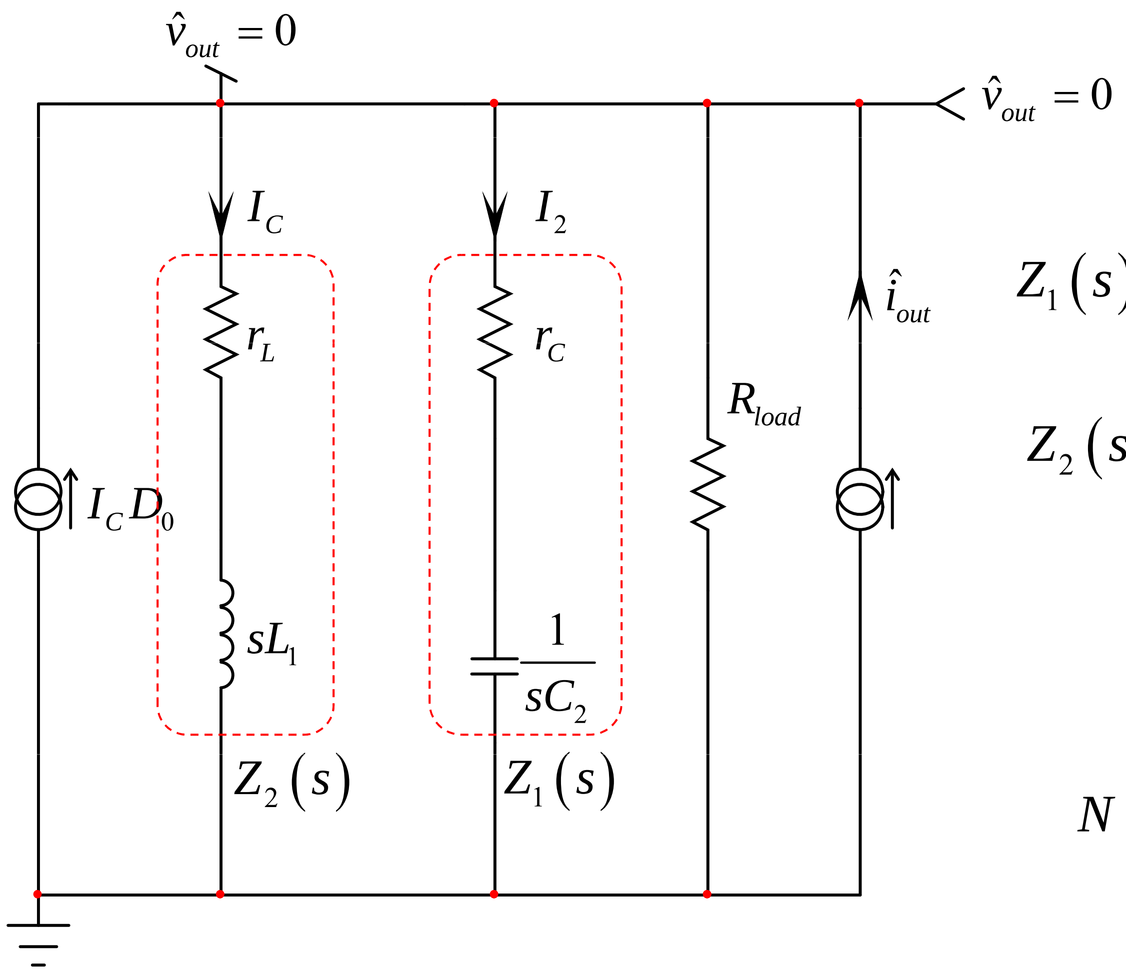
$$I_C = 0 \longrightarrow I_C D_0 = 0 \longrightarrow V_{out} = I_{out} R_{load}$$

There is a response: 2 zeros

Output impedance

Two Zeros in the Left Half-Plane

- The inductor contributes a zero with r_L



$$Z_1(s) = r_C + \frac{1}{sC_2} = 0 \longrightarrow \omega_{z_1} = \frac{1}{r_C C_2}$$

$$Z_2(s) = r_L + sL_1 = 0 \longrightarrow \omega_{z_2} = \frac{r_L}{L_1}$$

$$N(s) = \left(1 + \frac{s}{\omega_{z_1}}\right)\left(1 + \frac{s}{\omega_{z_2}}\right)$$

Output impedance

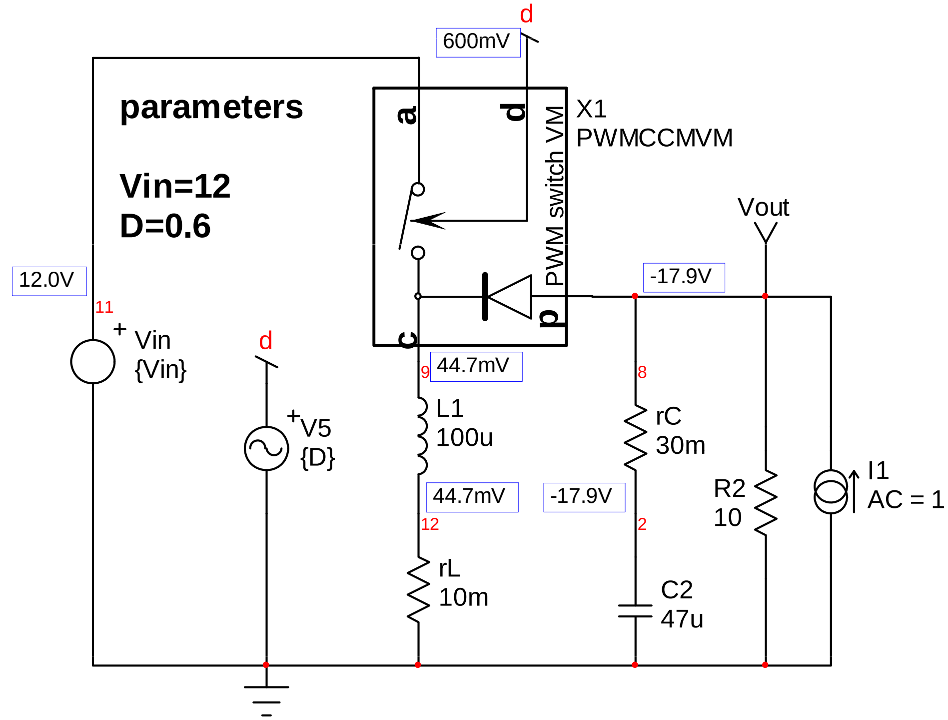
Final Transfer Function

- The transfer function is immediate

$$Z_{out}(s) \approx R_0 \frac{(1+sr_C C_2)\left(1+s\frac{L_1}{r_L}\right)}{1+\frac{s}{\omega_0 Q}+\left(\frac{s}{\omega_0}\right)^2}$$

$$\omega_0 \approx \frac{1-D}{\sqrt{L_1 C_2}}$$

$$Q \approx (1-D) R_{load} \sqrt{\frac{C_2}{L_1}}$$

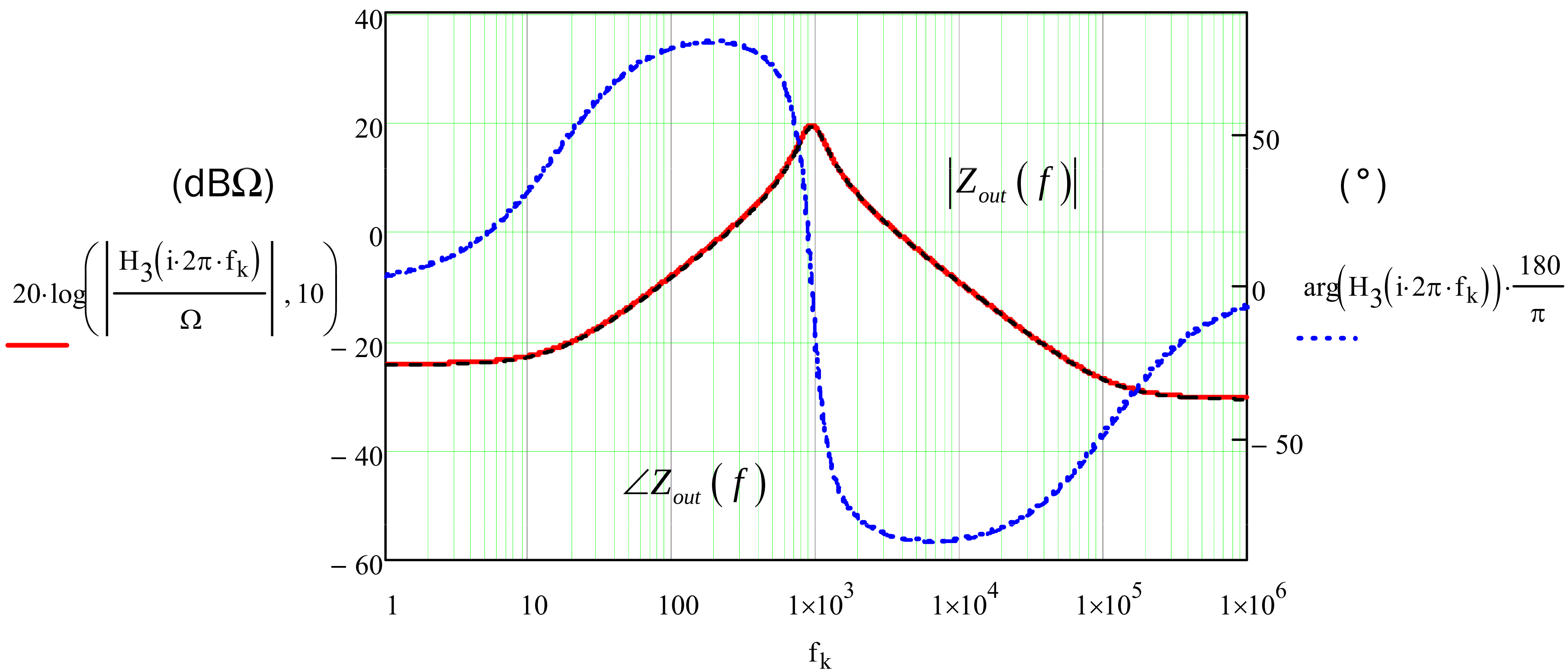


Large-signal PWM switch model

Output impedance

SPICE and Mathcad® Plots

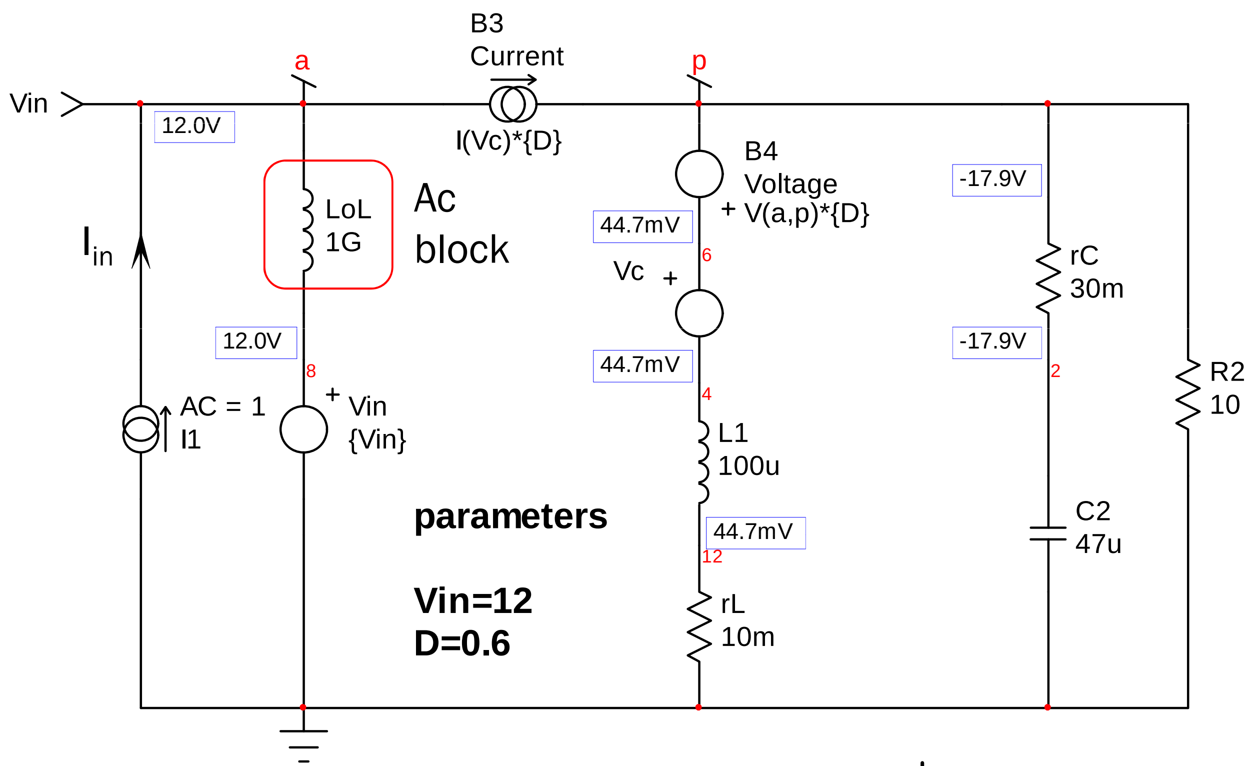
- Curves superimpose: transfer function is correct!



Output impedance

Buck-Boost Input Impedance

- LoL lets you ac-sweep the input to have Z_{in}



response $\longrightarrow$

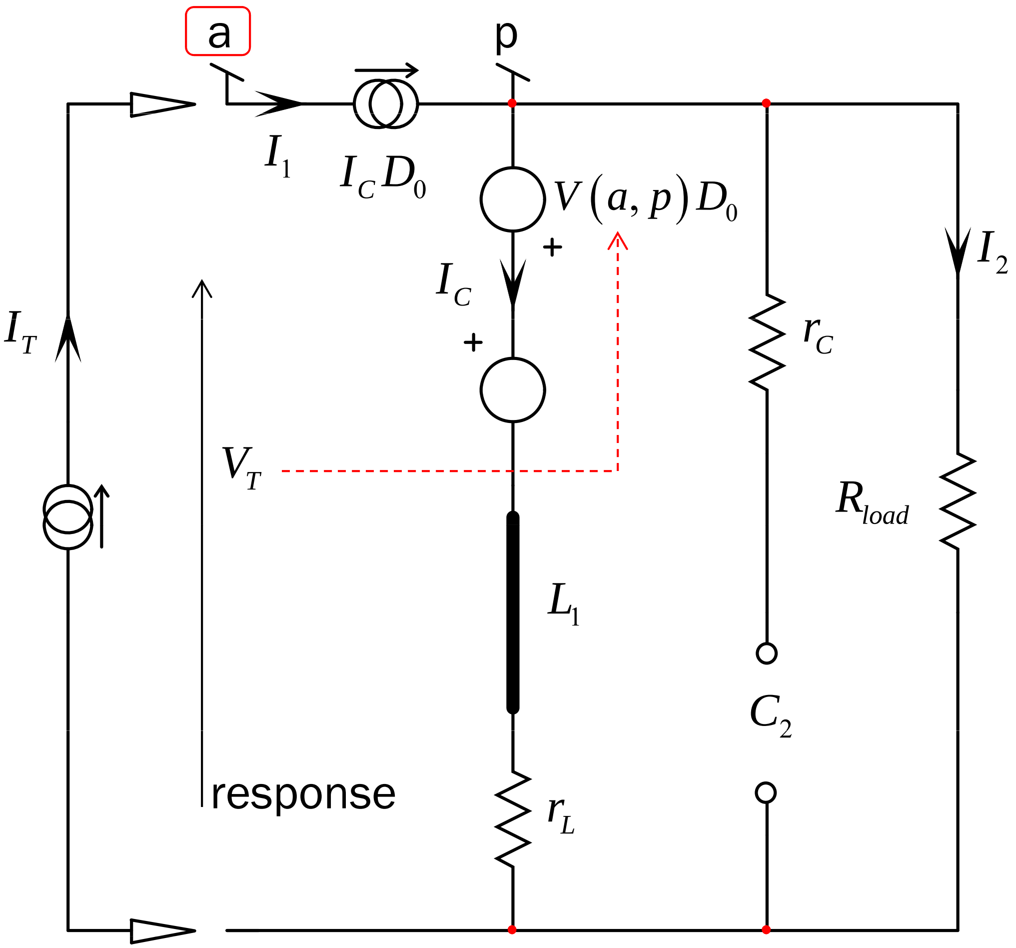
excitation $\longrightarrow$

$$\left.\frac{V_{in}(s)}{I_{in}(s)}\right|_{\hat{d}=0}$$

Input impedance

Input Resistance for $s = 0$

- Open capacitor and short the inductor



$$I_C D_0 = I_C + I_2 \qquad V_{(p)} = I_2 R_{load}$$

$$I_2 = I_C (D_0 - 1) \qquad V_{(p)} = I_C (D_0 - 1) R_{load}$$

$$I_C = \frac{V_{(p)} + V_T D_0 - V_{(p)} D_0}{r_L}$$

Substitute $V_{(p)}$

$$I_C = \frac{D_0 V_T}{R_{load} D_0^2 - 2R_{load} D_0 + R_{load} + r_L}$$

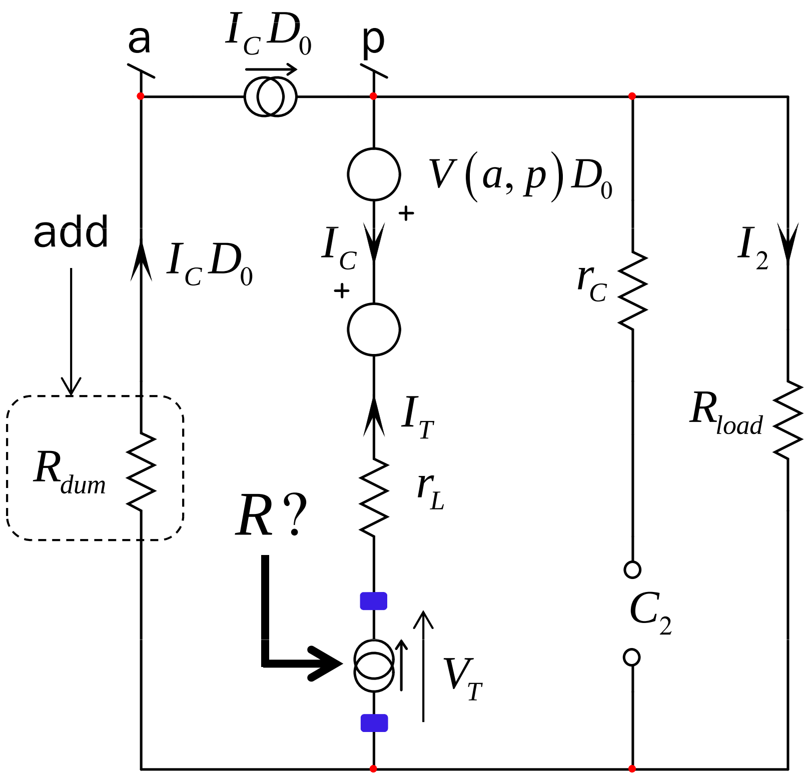
$$I_T = D_0 I_C$$

$$\frac{V_T}{I_T} = R_0 = \frac{(1 - D_0)^2 R_{load} + r_L}{D_0^2}$$

Input impedance

Determine the Denominator, τ_1

- ❑ Turning the excitation off changes the structure
  - ❖ You cannot reuse $D(s)$ and node (a) is dangling



$$V_{(a)} = D_0 I_T R_{dum} \quad I_2 = I_C D_0 + I_T \xrightarrow{I_C = -I_T} I_2 = I_T\left(1 - D_0\right)$$

$$V_{(p)} = I_2 R_{load} = I_T\left(1 - D_0\right) R_{load}$$

$$V_T = I_T\left(r_L + R_{dum} D_0^{\ 2} - \left(1 - D_0\right) R_{load} D_0 + \left(1 - D_0\right) R_{load}\right)$$

⇩ Rearrange

$$\tau_1 = \frac{L_1}{r_L + \left(1 - D_0\right)\left(R_{dum}\dfrac{D_0^{\ 2}}{1 - D_0} + R_{load}\left(1 - D_0\right)\right)}$$

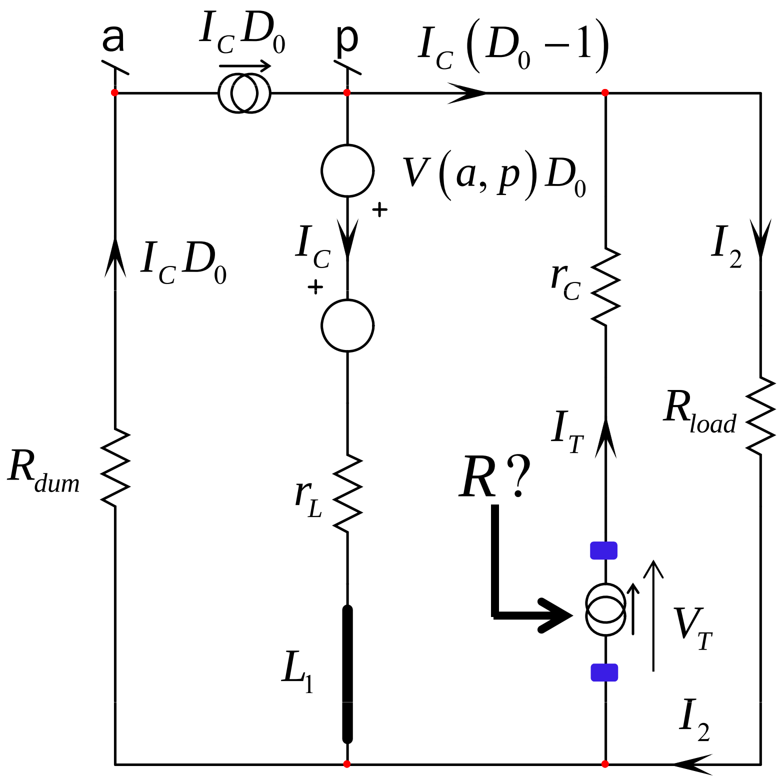
$$\tau_1\Big|_{R_{dum}\to\infty} = \frac{L_1}{\infty} = 0$$

- ❑ Install a dummy resistance to build a dc path

Input impedance

Determine the Denominator, τ_2

- Short inductor L_1 and look into C_2's terminals



$$\tau_2 = \left(r_C + R_{load}\right)C_2$$

$$V_{(a)} = -D_0 I_C R_{dum} \qquad V_{(p)} = \left[I_C\left(D_0 - 1\right) + I_T\right]R_{load}$$

$$V_{(p)} = V_T - I_T r_C \qquad I_C = \frac{V_{(p)} + V_{(a)}D_0 - V_{(p)}D_0}{r_L}$$

Substitute and rearrange I_C

$$I_C = \frac{I_T R_{load}\left(1 - D_0\right)}{R_{load} + r_L - 2D_0 R_{load} + D_0^2\left(R_{dum} + R_{load}\right)}$$

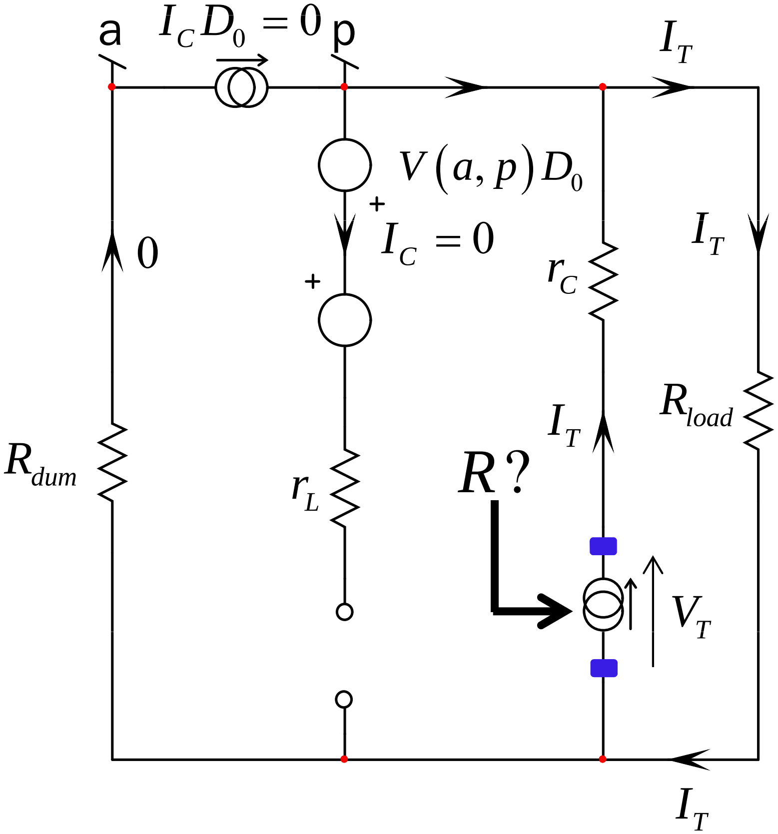
$$V_T - I_T r_C = \left[I_C\left(D_0 - 1\right) + I_T\right]R_{load}$$

$$R = \frac{V_T}{I_T} = r_C + \frac{R_{dum}\left(R_{load}D_0^2 + \frac{R_{load}r_L}{R_{dum}}\right)}{R_{dum}\left(\frac{R_{load} + r_L - 2D_0 R_{load} + D_0^2 R_{load}}{R_{dum}} + D_0^2\right)} \underset{R_{dum}\to\infty}{\approx} r_C + R_{load}$$

Input impedance

Determine the Last Term, τ_2^1

- Open the inductor and look through C_2's terminals



As L_1 is open, current I_C is zero

$$\tau_2^1 = C_2\left(r_C + R_{load}\right)$$

$$b_2 = \tau_1\tau_2^1 = \frac{L_1}{\infty}\cdot C_2\left(r_C + R_{load}\right) = 0$$

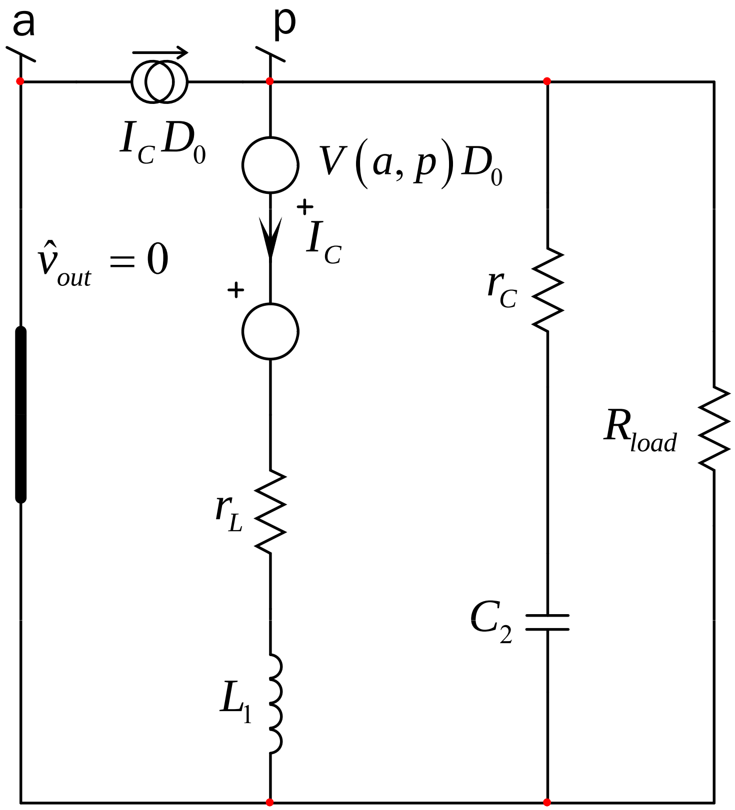
$$D(s) = 1 + b_1 s + b_2 s^2 = 1 + s\left(r_C + R_{load}\right)C_2$$

Input impedance

Null the response for the denominator

- Short the current source for a null in the response
- Structure returns to its original state: use D for N!

Determined for Z_{out} and H



$$N(s) = 1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2$$

$$\omega_0 \approx \frac{1-D}{\sqrt{L_1 C_2}} \qquad Q \approx (1-D) R_{load} \sqrt{\frac{C_2}{L_1}}$$

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Final Transfer Function

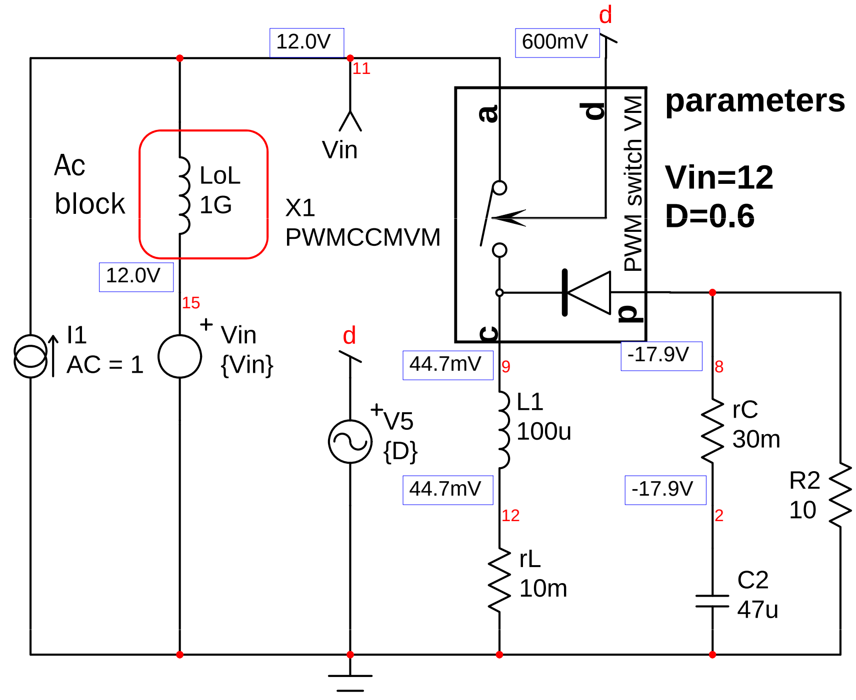
- The transfer function is immediate

$$Z_{in}(s) \approx R_0 \frac{1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2}{1 + s\left(r_C + R_{load}\right) C_2}$$

$$\omega_0 \approx \frac{1-D}{\sqrt{L_1 C_2}}$$

$$Q \approx (1-D) R_{load} \sqrt{\frac{C_2}{L_1}}$$

$$R_0 = \frac{(1-D_0)^2 R_{load} + r_L}{D_0^{\,2}}$$

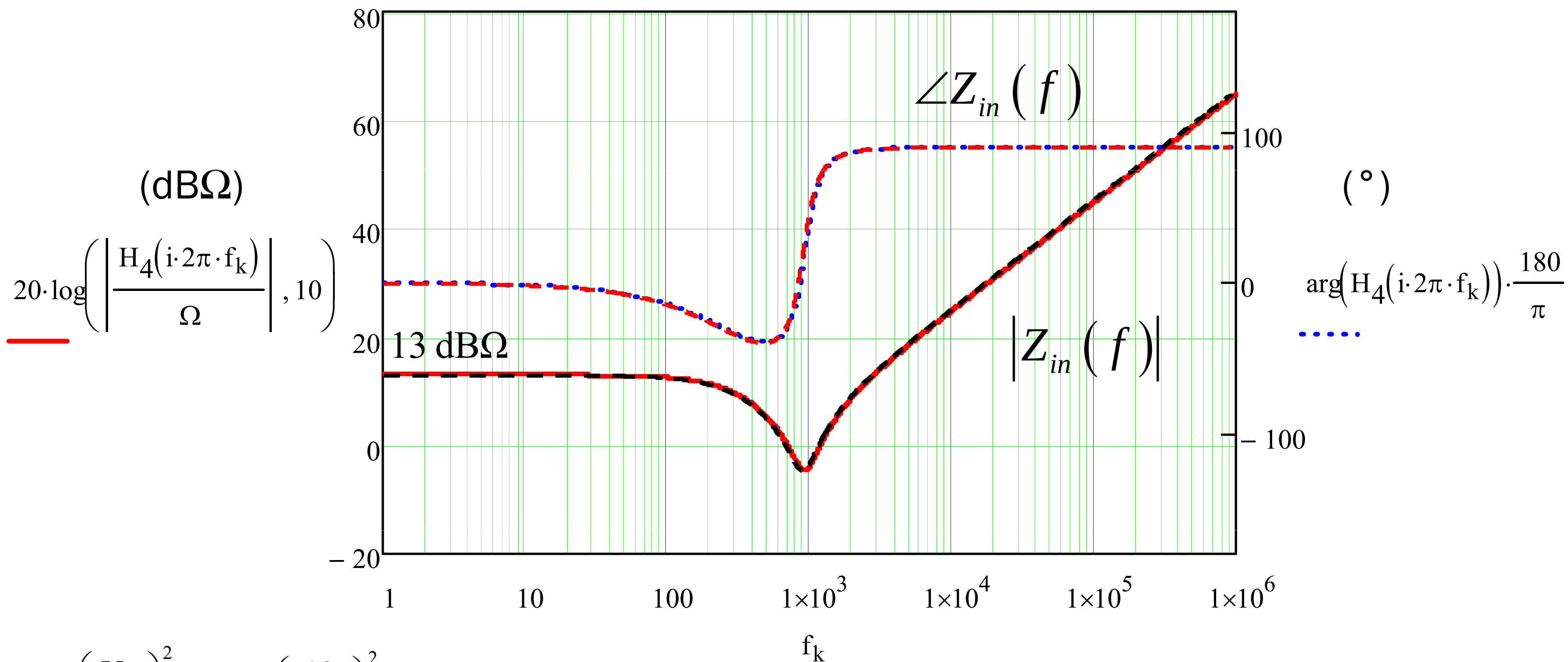


Large-signal PWM switch model

Input impedance

SPICE and Mathcad® Plots

- Curves superimpose: transfer function is correct!



$$R_0 = \left(\frac{V_{in}}{V_{out}}\right)^2 R_{load} = \left(\frac{12}{17.9}\right)^2 \times 10 = 4.49\ \Omega \rightarrow 13\ \text{dB}\Omega$$

Input impedance

References

- ❑ Middlebrook R.D. “Null Double Injection and The Extra Element Theorem”, IEEE Transactions on Education, Vol. 32, NO 3, August 1989
- ❑ R. D. Middlebrook, V. Vorpérian, J. Lindal, “The *N* Extra Element Theorem”, IEEE Transactions on Circuits and Systems, vol. 45, NO. 9, September 1998
- ❑ V. Vorpérian, “Fast Analytical Techniques for Electrical and Electronic Circuits”, Cambridge University Press, 978-0-52162-442-8, 2002
- ❑ C. Basso, “Linear Circuit Transfer Functions: A Tutorial Introduction to Fast Analytical Techniques”, Wiley IEEE-press, May 2016

Conclusion

- ❑ Plotting a transfer function is easy with nowadays tools
- ❖ You have no insight on what affects poles or zeros
- ❑ Analytical analysis is important but the form matters
- ❑ A *low-entropy* expression unveils contributors to poles/zeros
- ❑ FACTs naturally lead to *low-entropy* expressions
- ✓ Break the circuit into simple schematics
- ✓ Determine time constants in each configuration
- ❑ Small-signal analysis makes extensive use of FACTs
- ❑ SPICE and Mathcad are useful instruments to track errors
- ❑ Becoming skilled with FACT requires practice and tenacity!